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THESIS

PILOT STUDY ON THE APPLICABILITY OF VARIANCE REDUCTION TECHNIQUES TO THE SIMULATION OF A STOCHASTIC COMBAT MODEL

by

Curtis Smith

September 1987

Thesis Advisor: D. P. Gaver

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Prof. D. P. GAVER

22b TELEPHONE *(Include Area Code)*
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A Pilot Study on the Applicability of Variance Reduction Techniques to the Simulation of a Stochastic Combat Model

by

Curtis Smith
Captain, United States Army
B. S., United States Military Academy, 1978

Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

This thesis investigates the applicability of VRTs to the simulation of stochastic combat models. Ways of measuring the efficiency of a VRT are explored. Antithetic variates and stratified sampling are applied to the simulation of a trivariate Markovian combat model. Means of programming the antithetic variates and stratified sampling to reduce the inherent variability of uncertainty in the output data of the model are illustrated. Response surface regression models are used to characterize the performance of the antithetic variates and stratified sampling in the Markovian combat model.

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I. INTRODUCTION

A. GENERAL

War policies and plans for military operations made during peacetime are significant for the mission accomplishments of combat operations conducted during wartime. Since experimentation with real combat is infeasible, military analysts use stochastic combat simulation models to study the effects of policy making on combat operations. The analysts' inferences drawn from the results of these models are important to the decision maker since he has to use them to make the same decisions about military operations as he would if he could experiment with real combat itself. The output data from these models are realizations of random variables distributed around the values of the parameters of interest, or the models' true characteristics, so the analysts can only estimate these parameters with error. The magnitude of error for each estimate can be measured in terms of precision or the variance of the estimate: if the estimate is unbiased then the smaller the variance, the greater the precision, and the smaller the error. Since the decision maker wishes to make decisions that are based on estimate(s) with a quantifiable error bound, the analysts may find it possible to apply specific statistical techniques to measure and control the variance of the estimate(s) to obtain a prescribed or at least quantifiable level of precision.

The analyst's capability of estimating a parameter of interest with high precision depends on the extent to which he is able to control the sample variance. When the analyst uses the mean value of the sample output data as the estimate of a parameter of interest and when individual samples are independent, the coefficient of the variance of the estimator is reduced by a factor of 1/n where n is the sample size of the output data. A large sample size yields an estimate with a small variance and high precision. Multiple replications to obtain a large sample size in complex stochastic models can be prohibitively expensive in terms of resources like money, internal computer time, computer storage space, etc. This is especially true for large-scale, complex stochastic combat simulation models, which often require hours rather than minutes for a single computed replication. Since available computer time is a compelling constraint on military studies competing for scarce resources, the analyst is usually given an allocated amount of time to simulate his model. This specified amount of time may affect a desired level of precision of the estimate(s) that the analyst wishes to obtain from the simulation. Since the analyst can only execute a fixed number of replications within this block of time, the sample size (number of replications) may not be large enough to achieve a variance small enough to give the analyst an acceptable precision for the estimate(s). Hence, the analyst must either accept the particular level of precision and error associated with such variance or apply other specific statistical techniques which are more likely to produce a smaller variance, and hence a level of precision with which he can feel more comfortable.

An economizing scheme in simulation to reduce the variance of the estimator is to intentionally distort, control, and modify the random properties of the input variables in the simulation model. The output data resulting from the manipulation of these random numbers are random variables which are designed to be much closer together and more closely distributed around the true value of the model's parameter of interest than is the case with simple random sampling. A sample distribution resulting from such a variance-reducing scheme has the same mean value but a potentially smaller variance than the distribution of the sample without the usage of this scheme. The different techniques for doing this scheme are called Variance

Reduction Techniques (VRTs). The effects of certain of these, when applied to a combat model, are the subject of this thesis.

B. BACKGROUND

VRTs were initially used to evaluate multi-dimensional integrals. They have since been applied to small Monte Carlo simulation problems but have not been extensively utilized in large complex stochastic simulation models. The utilization of these variance-reducing techniques in real-world combat simulation models is even less common. Consequently, limited examples of applications of VRTs in these simulation models are found in the literature. The major reason for this is because the performance of the VRTs is suspected to be uncertain and unpredictable. The analyst has no guarantee that the usage of VRTs will work all the time. Futhermore, he has no way to know beforehand how much variance reduction he will get from the application of VRTs whenever they are effective. However, VRTs, in our opinion, promise to be powerful and effective tools in simulation if the issues of their performance in specific simulations are understood. In this section we will describe the effect that they can have on simulation studies. In Chapter V we illustrate their effectiveness for a particular combat model.

The effectiveness of a VRT may be measured by the relative efficiency of the simulation in obtaining estimate(s) with the utilization of this scheme, to the efficiency of a simulation under the same conditions without the VRT. Efficiency as Handscomb (1969,p. 253) defines it is

$$\text{Efficiency} = 1 / (\text{Variance} * \text{Work}). \qquad (1)$$

Here "Work" generally refers to computing time. According to Handscomb, variance reduction succeeds if the VRT increases efficiency. From Equation 1, we see that a decrease in variance and/or work will increase efficiency. Hence, variance reduction in simulation is more than solely a decrease in the variance of the mean of the estimators. Handscomb(1969,p. 253) calls a technique variance-reducing if it "reduces the variance proportionately more than it increases the work involved" or "does not reduce the variance at all in the usual sense, provided that it saves work." The work involved in attaining estimates by simulation has many attributes. Hammersley and Handscomb(1964,p. 22) suggest that the number of simulation runs epitomizes this work. However, we can easily measure this same work in terms of computing cost or/and simulation time. For it is the availability of these factors that ultimately determine the precision of the estimators. Hence, an effective VRT may not only produce more precise estimates but also economize the time and costs associated with the simulation to obtain the level of precision for those estimates. The efficiency of VRTs will be discussed more fully in Chapter III.

This potential saving in computer time has stimulated the interest of the United States Army Concept Analysis Agency (CAA) in the utilization of VRTs. CAA has studied the effectiveness of a VRT in two of its larger and more complex simulation models. The results of these studies were mixed (Johnson, Bates, and Graham, 1985). CAA therefore recommended the continuation of the studies to investigate the applicability of VRTs to reduce the inherent variability in large, complex, stochastic combat simulation models.

C. PURPOSE AND OBJECTIVE

The purpose of this thesis is to provide additional insight into the applicability of VRTs to stochastic combat models, and to provide a base for future studies in the application of VRTs to large-scale, real world, stochastic combat simulations. The objectives of this thesis are plainly to identify those VRTs that are applicable, and then to exhibit their performance in the applications to a class of stochastic combat simulation models. The question to be answered is: "Can VRTs be identified that are consistently effective for reducing simulation time and cost?"

D. PROBLEM

The problem for this thesis is to increase the efficiency of a stochastic combat simulation model utilizing VRTs in terms of (1) increased precision of the model's estimates for an allocated amount of simulation time, and (2) reduced computer time for a predetermined level of precision.

E. APPROACH

In his doctoral dissertation, Andréasson (1972) showed that variance reduction in queuing systems is influenced by (i) the transformation of random numbers, (ii) the structure and parameters of the simulation model, and (iii) the choice of the model response quantity. Condition (i) is an attribute of the VRTs. Conditions (ii) and (iii) are characteristics of the model. To solve the problem stated above, we investigate the effects of the parameters of a stochastic combat model, described in Chapter IV of this thesis, on variance reduction. We then use those results to formulate our approach to increase the efficiency of this model in terms stated in the problem above.

F. ORGANIZATION OF THIS THESIS

This thesis is organized into 6 chapters. Chapter I is the Introduction chapter. Chapter II reviews the literature of VRTs in simulation. Chapter III discusses ways of measuring the efficiency of a VRT and explores the tradeoffs of measuring for increased precision of estimation and reduced computer time. Chapter IV is concerned with the simulation of a stochastic combat model and the programming for variance reduction in the simulation model. Chapter V deals with the applicability and performance of VRTs in the simulation model. In Chapter VI, we make conclusions about the applicability of VRTs in stochastic combat models and provide recommendations about their use in larger and more complex, stochastic models that are used to study real-world combat systems.

II. REVIEW OF LITERATURE

A. INTRODUCTION

The VRTs that we use to solve the problem stated in Chapter I of this thesis are antithetic variates and stratified sampling. But first we review the literature of variance reduction in simulation. This chapter concentrates on the practical applications of VRTs in simulation models. We present a brief summary of works of scholars and experts on this subject. We then describe the basic concepts of two VRTs that we feel are applicable to large-scale, complex, stochastic combat simulation models. It is these two VRTs whose performances we later exhibit in the combat model in this thesis.

B. SUMMARY OF PREVIOUS WORKS

In the last 15 years interest in VRTs in simulation has stimulated much activity on this topic in the Operations Research community. This section does not comprehensively review all works that have been written in the literature, but it presents a brief overview of the utilization of VRTs in simulation. The purpose of this section is to summarize some of the studies of the general applicability of VRTs in simulation.

Hammersley and Handscomb (1964) reviewed many of the simplest ideas of variance reduction in simple Monte Carlo problems as they can be applied in the fields of Mathematics and Physics. Their most easily understood examples and outstanding successes were the evaluations of integrals and applications to particle physics. Handscomb (1969, p.252-262) later suggested that VRTs be adapted to simulation. He acknowledged difficulties in predicting the effectiveness of the techniques in particular situations, but he did propose, in practice, "... to proceed by more or less inspired trial and error, learning by experience which tools serve one best [or which techniques are effective]." He also stated that it may be much harder to tell how much variance reduction may occur in large and complicated simulation problems. These issues remain major concerns for one using VRTs in large, complex, stochastic simulation models.

Moy (1969, pp. 263-288) adapted several VRTs to simulation and investigated their applicability to queuing systems. He concluded that VRTs were indeed capable of working in the simulation of queuing systems. Kleijnen (1974, Ch. III), who has written probably the most comprehensive and most referenced documentation on the subject of VRTs in simulation, discussed the relevant differences between sampling in

Monte Carlo problems and sampling in stochastic simulation models. He showed that VRTs may be adapted to accommodate these differences. Kleijnen also presented a detailed description and critical appraisal of six techniques so devised for utilization in the simulation of large complex systems. These VRTs are stratified sampling, importance sampling, selective sampling, control variates, antithetic variates, and common random numbers. These six sampling techniques have become the most well-known and popular VRTs in the literature.

Other less-known VRTs, however, have been applied to simulation. McGraft and Irving (1974) survey some 18 different techniques for implementation in large scale simulation problems. McGraft and Irving include a comprehensive listing of the characteristics, advantages and disadvantages, and criteria for applicability to large simulation models, and demonstrate the effectiveness of several of these techniques with a military simulation application.

Many other articles and papers have been written on the subject of VRTs. There are too many of them to list in this thesis, but the survey ranges from specific techniques to more general methods in simulation experimentation. Some of the most recent papers written about the general applicability of VRTs in simulation are Nelson (1985) and Cheng (1986). Textbooks that illustrate the applications with simple but excellent examples of variance reduction in simulation are Gaver and Thompson (1973, Ch. 12), Fishman (1978, Ch. 3), Law and Kelton (1982, Ch. 11), Morgan (1984, Ch. 7), and Bratley, Fox, and Schrage (1987, Ch. 2).

C. DESCRIPTION OF VRTs USED IN THIS THESIS

Moy (1969, p. 263-288) experimented with antithetic variates and stratified sampling and showed that they are indeed capable of significantly decreasing variability in the simulation of simple queuing problems. Likewise, we wish to achieve similar results when we apply them to the simulation of our stochastic combat model. We do this in Chapter V. In this section, we discuss the underlying conditions and fundamental concepts in the applicability of antithetic variates and stratified sampling in simulation.

1. Antithetic Variates

The method of antithetic variates is relatively well-known in the literature of variance reduction in simulation (Kleijnen, 1974). It is one of the most useful VRTs because of its simplicity and general applicability. When the method of antithetic variates is used, the sampling process is modified by the manipulation of random numbers. A simulation run produces a response from the original sequence of random numbers {r}; then, a second simulation run produces an antithetic response from the sequence of the complementary random numbers {1-r}. The average of the two responses is an observation on the sample output data of the stochastic simulation model. The mean value of this sample is estimated as the parameter of interest.

The variance of this estimate is reduced if the responses of the first and antithetic runs of each replication are negatively correlated. Besides the interchanging of the random numbers in each run, two other conditions must occur to produce negative correlation between the runs. First, each response must be a monotonic function of its respective random number stream; that is, large values in each stream of random numbers should have an opposite effect on the response than the small values, and vice versa. The second condition is that the responses to the events in the first run must be synchronized with the responses to the events in the antithetic run. Synchronization, defined by Kleijnen (1974, p. 193), occurs

> ...if the i'th random number r_i generates [in the stream of the first run] a particular event (e.g., arrival of customer j) then in the antithetic run $(1 - r_i)$ should generate the same event (i.e., not the arrival of customer j' where $j' \neq j$ and not a service time).

If the antithetic variates methodology, coupled with required conditions, is designed into the simulation, then the average of the two negatively correlated responses will tend to produce an estimate with a high degree of precision. That is, if by chance (r) yields a response above the value of the true parameter of interest, then (1-r) should yield a response below the value of the true parameter. When these responses are averaged, the deviations between the responses and the true parameter approximately offset each other resulting in relatively small net variability in the output data. This idea can be shown mathematically. Let X_1 be the response of the first run; X_2, the response of the antithetic run; and Y, the average of X_1 and X_2.

$$Y = (X_1 + X_2) / 2$$

$$VAR_Y = 1/4 * (VAR_{X1} + VAR_{X2} + 2 * COV_{X1,X2})$$

$$= 1/2 * (1 + CORR_{X1,X2}) * VAR_{X1}$$

Clearly, a negative $CORR_{X1,X2}$ reduces VAR_Y. If $CORR_{X1,X2}$ equals, or is close to, -1, then the VAR_Y is mathematically zero or very close to it. Hence, the antithetic sampling is designed in simulation models so that the correlation between the pair of responses is as close to -1 as possible.

Monotonicity and synchronization must be designed into a simulation program for a particular model . Kleijnen (1974), Law and Kelton (1982), and Bratley, Fox, and Schrage (1987) are excellent references that discuss ways to do this. We discuss our design to achieve these two conditions for antithetic variates in our model in Chapter IV. As stated before, the method of antithetic variates is simple to implement and requires little to no extra computer time. Because of simplicity of this VRT, examples of its applications are illustrated in nearly every textbook that considers the subject of VRTs.

2. Stratified Sampling

The stratified sampling technique, discussed in this section and applied to the simulation model in chapter V of this thesis, is a different version of the stratified sampling that Moy, Kleijnen and other experts on VRTs have adapted to simulation. Handscomb (1969, p. 261) calls this particular version of stratified sampling another form of antithetic sampling. Andreasson (1972, p. 6) refers to it as an antithetic transformation. Gaver and Thompson (1973, pp. 585-586) name it stratification extending an antithetic idea. It is indeed stratification in that the sampling process is modified so that the range of random numbers is divided into two or more strata from which the simulation runs produce responses. It has the antithetic flavor in that the responses in all strata are averaged together to get an observation which is part of the sample output data. This technique is also similar to antithetic variates in that its estimator is an average of correlated responses (Gaver and Thompson 1973, p. 586). Likewise, this estimator tends to have a smaller variance.

In our review of this technique, we saw no necessary conditions, like those for the antithetic variates, for this technique to be successful in simulation. The design of stratified sampling into our simulation model in Chapter IV is similar to that one given in Gaver and Thompson (1973, p.586).

D. SUMMARY

An abundant amount of material has been written on the subject of variance reduction. Techniques used to reduce the variance in Monte Carlo problems have been adjusted to do the same in simulation models. The applications of VRTs in simulation have been illustrated in queuing systems and simple textbook problems but successful applications to larger, more complex, real-world stochastic simulation models have not been so amply reported. There is no guarantee that VRTs will work spectacularly for every situation in the simulation, and when they do work it is necessary to estimate the magnitude of the variance reduction. Pilot tests are encouraged to help resolve these issues. Antithetic variates and modified versions of stratified sampling are two of the more simple and easily employed VRTs and will be applied to a stochastic combat model in Chapter V.

III. EFFICIENCY OF VARIANCE REDUCTION

A. INTRODUCTION

In the last chapter we reviewed some studies that involved VRTs. In this chapter we discuss the problem of measuring the efficiency of a VRT. Comparing variances of a parameter of interest obtained from the simulations with and without the use of a VRT respectively, on an ordinal scale, may reveal if the VRT works, but it provides little information about how well the VRT works. Clearly, a quantitative measure is more desirable. Therefore, the manner or scale on which the efficiency of a VRT is measured should provide as much information as possible on the performance of a VRT. In particular, it should provide at least some base to answering the following questions:

(i) "Does the VRT work?"

(ii) "If so, how great is the variance reduction in terms of increased precision for estimating the parameter of interest?"

(iii) "How great is the variance reduction in terms of simulation time saved for estimating the parameter of interest?"

(iv) "What are the tradeoffs, if any, between the potential increase in precision and the economy of simulation time when applying VRTs?"

In the next section we examine two methods that are usually used in the literature to measure the efficiency of a VRT. We evaluate them in terms of how well they answer the questions above. In the third section, we offer a third alternative which is a hybrid of the two previous methods for measuring the efficiency of a VRT. This third method, we think, answers all four questions above and is used to measure the efficiencies of the antithetic variates and the stratified sampling techniques whose performance is exhibited in this thesis. In the fourth section of this chapter we show how to use the third method of measuring the efficiency of a VRT to obtain the tradeoffs between increased precision and reduced simulation. The last section is a summary of this chapter.

B. ASSESSMENT OF VARIANCE REDUCTION

In the literature the efficiency of a VRT is usually measured by (1) a decrease in the variance (Method #1) or (2) the relative efficiency of a simulation to obtain an estimate using a VRT to the efficiency of the simulation using no VRT (Method #2). Henceforth, we refer to a simulation without the use of a VRT as crude simulation.

Method #1 is well defined in Kleijnen (1974, pp. 106-107). Kleijnen uses this method by defining the efficiency of a VRT as a percentage of reduction in varianc

$$\text{Method \#1} = (\text{Var}_0 - \text{Var}_1) / \text{Var}_0) * 100\% \qquad (2)$$

where Var_0 and Var_1 are variances obtained in the same amount of simulation time for crude simulation and simulation applying a VRT respectively. The measure of efficiency of a VRT which Kleijnen introduces may be interpreted as that portion of the variance which is not achieved by crude simulation but is obtainable in the same amount of simulation using a VRT. The sign of this portion determines whether the VRT increases or decreases the precision; a positive sign reveals an increase and a negative sign, a decrease. The magnitude of the portion indicates how much of the precision is increased or decreased respectively. With this method we can also see that the VRT has an identical effect on reducing simulation time for a prescribed level of precision as it does on increasing precision. Method #1 provides answers to three of the questions stated in the last section, but it does not resolve the question of tradeoffs for increased precision and reduced time in a simulation using VRT.

McGrath and Irving (1974, p. 295) use Method #2 to measure the efficiency of a VRT. They initially used this method, shown as Equation (3), to equate the relative advantage gained in simulation time by using a VRT.

$$\text{Method \#2} = \mathrm{Var}_0/\mathrm{Var}_1 * (\text{Simulation Time}_0)/(\text{Simulation Time}_1) \quad (3)$$

where subscripts 0 and 1 represent crude simulation and simulation applying a VRT respectively. The relative efficiency that McGrath and Irving used to measure the efficiency of a VRT results in a factor by which the efficiency of a simulation is increased or decreased by using a VRT. If the value of this factor is greater than one, then the VRT works; otherwise, it does not. The magnitude of this reduction is the actual value of the factor. For example, if the value of the factor is 5, then the simulation applying the VRT can obtain an estimate in 1/5 the simulation time required by the crude simulation for the same precision level. Method #2 may be viewed either as the reduction in simulation time when both simulations are to achieve the same variance, or as an increase in precision when both simulations are run for the same amount of time. This method, like Method #1, answers only the first three questions proposed in the first section of this chapter.

C. THE HYBRID METHOD

Methods #1 and #2 measure increased precision at a fixed simulation time, or a reduced simulation time at a fixed level of precision. They do not, on the other hand, measure increased precision at a level of reduced simulation time, or vice versa; nor do they provide a means to explore such a possibility. In Chapter I we emphasize that variance reduction may increase precision and reduce simulation time. The efficiency of a VRT, in our opinion, should reflect both effects so that we can explore the tradeoff of any combination of precision and simulation time. Method #3 offers such possibility and answers all four questions in the introduction section of this chapter. It is a mixture of Methods #1 and #2. Method #3 has Kleijnen's idea of reduction in variance and McGrath and Irving's use of relative efficiency. We define the efficiency of a VRT as a relative efficiency (RE), as shown in Equation 4, and later define it in terms of increased precision (IP) and reduced time (RT).

$$\text{Method \#3} = \text{Efficiency}_1 \,/\, \text{Efficiency}_0 \tag{4}$$

where Efficiency_0 and Efficiency_1 are the efficiencies of the crude simulation and simulation applying a VRT respectively. The definition of the efficiency of a simulation, identified by Equation 1, is the inverse of the product of the sampling variance of the parameter estimate and the work. Henceforth, we equate work to simulation time, which is the total time of the simulation model to obtain a parameter of interest and a specified variance. Such time may be computed as n replications times T (average) simulation time per run. If k runs are in a replication, then simulation time equals the product of kn runs and T (average) simulation time per run. When these variables are substituted in Equation 1, the efficiency equation becomes Equation 5:

$$\text{EFFICIENCY} = 1 / (\text{Var} * k * n * T) \tag{5}$$

If we are to increase the efficiency of a simulation using a VRT, then we must attempt to decrease one of the parameters in Equation 2. The variable k runs per replication is a constant of the VRT. Specifically, the antithetic variates constant k is two; stratified sampling constant k is three in our study (it can be more); and for no VRT, the constant value of k is one. The variable T is model dependent; that is, its value depends on the input parameters of the model. Attempts to decrease this variable may be futile; futhermore Bratley, Fox, and Schrage (1987, p. 48) point out that relatively little can be done to decrease it. We discussed the relationship between the variance (Var) and replications (n) in Chapter I. They are ,in essence,the variables we wish to decrease. Throughout this thesis we interchange the phrases decrease in variance with increased precision and reduction in replications with reduced time (simulation). If we substitute Equation 5 into Equation 4, we get Equation 6. Note since T (average) simulation time per run is the same for both simulations, it is left out of the equation.

$$RE = Var_0/Var_1 * k_0/k_1 * n_0/n_1 \qquad (6)$$

If the RE value in this equation is greater than one, then the VRT successfully increases the efficiency of the simulation and is said to be strong; otherwise, it is said to be weak. A strong VRT decreases the variance so that precision is increased and simulation time is reduced. A weak VRT, on the other hand, does not decrease the variance as well as a strong VRT; in fact, a very weak (or subversive) VRT may increase the variance, which causes a reduction in precision and necessitates an increase in simulation time. In most simulation models a VRT may be strong for certain conditions and weak for other conditions. In this thesis, we look for such characterizations of the antithetic variates and the stratified sampling in the stochastic combat model we describe in the next chapter. In the next section, we define Equation(6) in terms of increased precision and reduced simulation time.

D. TRADEOFFS OF GAINING PRECISION AND SAVING TIME

Let us define IP as a decrease in variance,

$$IP = (Var_0 - Var_1) / Var_0 \qquad (7)$$

and RT as a reduction in simulation time

$$RT = (n_0k_0 - n_1k_1) / n_0k_0. \qquad (8)$$

Then

$$Var_1 = (1.0 - IP) * Var_0 \qquad (9)$$

and

$$n_1k_1 = (1.0 - RT) \text{ x } n_0k \qquad (10)$$

If we substitute Equations (9) and (10) into Equation (6), we get Equation(11).

$$RE = 1/(1.0 - IP) * 1/(1.0 - RT) \qquad (11)$$

Equation 11 defines the relative efficiency which we defines as Method #3 of measuring the efficiency of a VRT, in terms of increased precision and reduced time. This equation resolved the unanswered question identified as (iv) in the introduction section of this chapter. With this equation, we can examine any combination of IP and RT. For example, suppose we measure the efficiency of a VRT to have a RE value of 6 for the same amount of simulation time (Hint: RT=0). Substituting these values into Equation (11), we get IP = 5/6 or 83.3% increased precision.

Suppose we only need to increase the precision to 75% instead of 83.3%, then we can substitute the values for IP=3/4 and RE=6 (RE should not change) into Equation 11. We now get RT=1/3 (Note, we increase the precision 75% and reduce the simulation time 33.3%). Likewise, with RE=6 for the efficiency of the VRT, examples of other combinations are (IP=2/3,RT=1/2); (IP=1/2,RT=2/3); and (IP=0,RT=5/6). In fact, we may get any combination of (IP,RT) between 0 and 5/6. Note, however, if we want to increase the precision beyond 83.3% or 5/6, we will get an increase in simulation time. That is the tradeoff in terms of more increased precision. For example, we will have to increase the simulation time to 2/3 or 66.7% to accommodate an IP of 90% for a RE value of 6. In short, the information obtained from method #3 is that we can make an estimate more precise and save simulation time simultaneously.

E. SUMMARY

In the literature, there are generally two methods of measuring the efficiency of a VRT. Method #1 is a decrease in the variance; Method #2 is the relative efficiency of a simulation using VRT to crude simulation. Both methods may determine if VRT works in a simulation model. They also may indicate the magnitude of the variance reduction in terms of either increased precision for a fixed simulation time or reduced simulation time at a fixed level of precision. In this chapter, we introduced a third method of measuring the efficiency of a VRT. It is a hybrid between Method #1 and Method #2. Method #3 offers exploration into the tradeoffs of increasing precision and saving simulation time for any efficiency value of a VRT.

IV. SIMULATION OF A STOCHASTIC COMBAT MODEL

A. INTRODUCTION

In the last chapter we discussed how to measure the efficiency of a VRT to determine how much we may save in simulation time or/and how much we may increase the precision of a parameter obtained by crude simulation. In this chapter we show how we may apply VRTs to the simulation of a combat system. The combat model which we have chosen to simulate and to apply the VRTs is the BCD Markovian model developed in the doctoral dissertation of Abdul-Latif Rashid Al-Zayani (1986). A modified version of this model, formulated by Professor Donald P. Gaver, is in Appendix A. This stochastic model may seem very simple, but its simulation provides invaluable insights into the applicability of VRTs to stochastic combat model.

Beside being stochastic, the BCD Markovian model is also discrete and dynamic in nature; hence, it is a discrete-event simulation model. We refer those readers who want to know about the nature of discrete-event simulations to simulation textbook such as Morgan(1984), Law and Kelton(1982), or Bratley, Fox, and Schrage(1987). In this thesis, we describe the simulation of the BCD model in terms of discrete events. We describe, in detail, the crude simulation of the BCD model in the next section. In Section C, we show how we applied antithetic variates and stratified sampling to this simulation. We summarize the chapter in the last section.

B. CRUDE SIMULATION OF THE BCD MARKOVIAN MODEL

We discuss the crude simulation of the BCD model in four parts. First, we describe the combat scenario; second, we define the characteristics of the model; third, we explain the simulation of the combat process in the model; and finally, we discuss a FORTRAN simulation program written for the model.

1. The Combat Scenario

As part of an air defense command, a wing of aircraft defenders is responsible for defending an area against a hostile air attack from a group of bombers. When detection of an incoming threat occurs a flight of D defenders is launched to engage B bombers making the attack. When the two groups are within aerial combat range the defenders seek a one-to-one combat engagement with the bombers at a rate Φ. Only one free defender can engage a free bomber in combat; a bomber will generally attempt to avoid any engagement with a defender. A combat engagement lasts until either the bomber is killed or the defender is killed. A defender kills a bomber at a rate α, and a bomber kills a defender at a rate β. Hence, at any instant during the combat process, a defender is either free and searching to fight a bomber, fighting a bomber, or killed. A bomber is, likewise, either free and eschewing engagement with a defender, engaging in combat with a defender, or killed. The combat process is continued until either force is completely killed off or the duration of the combat period is terminated after T units of time.

2. The Characteristics of the BCD Model

Hartman (1985, p. 2-18) characterizes the structure of a combat simulation model as combat entities, attributes, and events. We use these characteristics to simulate the combat process of the BCD model in the next subsection. Combat entities in the BCD model are free bombers, free defenders, and combat engagements. Each entity has attributes that describe a combat scenario. For the bombers, the attributes are the number of bombers and the rate a bomber shoots down a defender; for the defenders, the number of defenders and the rate a defender shoots down a bomber; and for the combats, the number of combat engagements and the rate that a bomber and a defender engage in combat.

Law and Kelton (1982, p. 4) define an event in a discrete-event simulation as an occurrence which changes the state of the system. The BCD model has five events. The first event is the initialization of the air battle. The initialization event governs the initial battle conditions. The next three events, are the interim events in the combat process. These events are (1) a combat between a bomber and a defender, (2) a defender killing a bomber, and (3) a bomber killing a defender. The occurrence of an interim event changes the state of the combat process at time t. The state of the combat process of the BCD model is represented by the trivariate-Markov process $\{B(t),C(t),D(t);t>0\}$; where, B(t) is the number of free bombers at time t, C(t) is the number of combat engagements at time t, and D(t) is the number of free defenders at time t. As a Markov process, the combat process moves from state to state according to one-step transit probabilities that depend only on the current state. The fifth and last event in the combat process is the termination of the air battle. The termination event manifests the "end of the battle" conditions. The values of the state of the system at the occurrence of the termination event reflects the battle outcome. These values are the numbers of bombers and defenders that are alive at the end of this air battle. We will consider these values as the parameters of interest in the simulation of the BCD model.

3. Simulation of the Combat Process

We simulated the combat process by maintaining a "bookkeeping account" of the changes in the state of the combat process as the events occur. The process begins in the initial state $\{B(t),C(t),D(t);t=0\}$ with the initialization event being the commencement of the air battle. Henceforth, we let a value of $B(t)$ equal b, a value of $C(t)$ be c, and of $D(t)$ be d. The interim events change the value of the state of the combat process as following:

EVENT	STATE
New combat	$b-1,c+1,d-1$
Bomber kills Defender	$b,c-1,d+1$
Defender kills Bomber	$b+1,c-1,d$

The combat process spends $X(b,c,d)$ units of sojourn time in state (b,c,d) until another event occurs. The sojourn time $X(b,c,d)$ is a random variable distributed exponentially with mean

$$P(b,c,d) = 1/(\Phi BD + (\alpha + \beta)C)$$

for the time to the next event in the combat process of the air battle, given that at time t the state was (b,c,d). Equation 12 is this sojourn time. To derive this expression, we use the inverse transform method to obtain unit-mean exponential random variables, where V_j is the jth random number in the sequence of a stream of uniform random numbers. The inverse transform method is discussed in the simulation textbooks listed in the reference section of this thesis.

$$X(b,c,d) = - P(b,c,d) * \ln(V_j) \tag{12}$$

We use the value of Equation 12 to advance the simulated time of the air battle as indicated by Equation 13.

$$t = t + X(b,c,d) \tag{13}$$

The combat process moves to another state when another event occurs. The probability of a specific interim event occurring is governed by an embedded Markov chain whose transition probabilities are as follows:

EVENT	PROBABILITY
New Combat	$\Phi BD * P(b,c,d)$
Defender kills Bomber	$\alpha C * P(b,c,d)$
Bomber kills Defender	$\beta C * P(b,c,d)$

We again use the inverse transform method to obtain the conditions for an interim event to occur and to induce the change in the state of the combat process. V_j is the jth random number in the sequence of a different stream of random numbers. These conditions, events, and changes in the state of the combat process are listed below.

CONDITION	EVENT	STATE
$V_j \leq \Phi BD * P(b,c,d)$	New Combat	b-1,c+1,d+1
$V_j > \Phi BD * P(b,c,d)$ and $V_j < (\Phi BD + \alpha C) * P(b,c,d)$	Defender kills Bomber	b,c-1,d+1
otherwise	Bomber kills Defender	b+1,c-1,d

Thus, we (i) generate a uniform random number to choose which interim event has occurred, (ii) update the state of the combat process, (iii) generate another uniform random number and transform it to an exponential random variable to determine the unit of time until the occurrence of the next event and to advance the simulated time of the combat process. We repeat this procedure until the occurrence of the termination event. The termination event occurs when (1) all the bombers are killed ($B(t) + C(t) = 0$) or all the defenders are killed ($D(t) + C(t) = 0$) or (2) the time duration of the air battle has expired ($t \geq T$ units of time). At the end of the aerial battle, the combat process is in state (b,c,d) from which we can compute the number of live bombers B ($B(t) + C(t)$) and the number of live defenders D ($D(t) + C(t)$). These are values of random variables for one possible battle outcome.

4. The FORTRAN Simulation Program

We coded the crude simulation of the BCD model in FORTRAN. This FORTRAN program, consisting of a main program and four subroutines, is in Appendix B. The main program begins in an interactive mode. The program reads the values for the attributes of a combat scenario from the terminal and sends them to the BATTLE subroutine. BATTLE runs N replications of the combat process and returns the summary statistics of the outcome of N battles to the main program. The main program sends them to the STAT subroutine. STAT analyzes these battle statistics in terms of parameter estimates and returns the values of these parameter estimates to the main program. The main program then sends these values of parameter estimates to a formatted output file.

The BATTLE subroutine calls two subroutines UNIFOR and EXPON for the generation of U(0,1) random numbers. These two subroutines implement the congruential pseudo-random number generator

$$U_{i+1} = 16807U_i \bmod (2^{**}31 - 1) \qquad (14)$$

discussed and tested by Lewis and Orav (1985, Ch. V). UNIFOR generates a sequence of uniform random numbers for the selection of the occurrence of an interim event. EXPON generates a sequence of uniform random numbers for the computation of the unit of sojourn time in a state.

The STAT subroutine performs statistical output analysis for the simulation. It computes the means and variances of the sample distributions of live bombers and defenders. The sample means for bombers and defenders

$$\bar{B} = \text{SUM } B_i \ / \ N \qquad (15)$$

$$\bar{D} = \text{SUM } D_i \ / \ N \qquad (16)$$

are unbiased (point) estimators of E[B(t)] and E[D(t)] respectively. Similarly, the variances

$$VAR_{\bar{B}} = SUM\ (\ B_i - B)**2\ /\ N * (N-1) \qquad (17)$$

$$VAR_{\bar{D}} = SUM\ (\ D_i - D)^{*}*2\ /\ N \times (N-1) \qquad (18)$$

are unbiased estimators of VAR[E[B(t)]] and VAR[E[D(t)]] respectively (Larson, 1982).

C. PROGRAMMING FOR VARIANCE REDUCTION

In Chapter I, we noted that VRTs modify the sampling of random numbers. In this section, we discuss these modifications for the antithetic variates and stratified sampling in the simulation of the combat process. We describe the changes we made to the crude FORTRAN simulation program for the simulations using antithetic variates in Section 1 and stratified sampling in Section 2 respectively.

1. Antithetic Variates

We make changes to the subroutines BATTLE, UNIFOR, and EXPON of the crude simulation program to use the antithetic variates. The FORTRAN program for the BCD simulation model applying antithetic variates is in Appendix C. The BATTLE subroutine computes the values of the parameters for one replication as the average values of the battle outcomes from a pair of runs of the combat process. We obtain the values of the battle outcome from the first run by using a stream of uniform random numbers (U) and those of the second run by using a stream of complementary random numbers (1-U). Since we are attempting to decrease the variance of the estimates by inducing negative correlation between these two runs, we want to minimize this negative correlation. We, first, induce a negative correlation between the two runs by creating monotonicity between the random numbers and the values of the battle outcome within each run. We then minimize this negative correlation by synchronizing the sequences of random numbers (U) and the complement (1-U) (Bratley, Fox, Schrage 1987, p.47). Kleijnen (1974, p.187) shows that a random variable generated by the inverse transform approach is monotonic. Hence we have monotonicity in the simulation since we used the inverse transform method to generate the uniform random variables in the simulation of the BCD model.

Law and Kelton (1982, p. 352) indicate that the inverse transform approach also facilitates the maintenance of synchronization . With this method, we use only one uniform random number per sequence to obtain the desired random variable for each event in the combat process; as contrasted with other methods, like the rejection method, where we may use many random numbers to produce a single value for the desired random variable of the same events. Thus we initiate synchronization in the model when we use the inverse transform method; nevertheless, we must still preserve it.

We risk losing this synchronization in the BCD model because the number of interim events occurring in the combat process per run is a random variable. Hence the number of events occurring in the combat process in the first run may not be the same as the number of events occurring in the combat process in the antithetic run. Consequently, the number of random numbers needed in the antithetic run generally differs from that required in the first run. This phenomenon leads to the random number $\{U_j\}$ in the first run not being synchronized with the random number $\{1-U_j\}$ of the antithetic run (Kleijnen 1974, p. 193). In other words, the complement of the jth uniform random number $\{1-U_j\}$ is not used for the jth event in the combat process in the antithetic run. We are not able to control the random number of interim events in the combat process, but we can manage the way in which UNIFOR and EXPON generate uniform random numbers so that synchronization is maintained between the pair of runs per replication.

We used the suggestions of Law and Kelton (1982, p.352) and Bratley, Fox, and Schrage (1987, p.47) to maintain the synchronization that the inverse transform method has initiated in the BCD simulation model. We modify subroutines UNIFOR AND EXPON to generate separate streams of random numbers {U} and complements {1-U} before simulating a pair of runs of the combat process. When the subroutine BATTLE calls UNIFOR and EXPON, it receives from each a two-dimensional array of random numbers, where the first column contains the stream {U} and the second column contains the stream {1-U}. Hence, we use the first column for the first run and the second column for the antithetic run. This approach guarantees that if the jth event in the first run uses $\{U_j\}$, then the jth event in the antithetic run will use $\{1-U_j\}$. We do waste some of the random numbers in the arrays, but we do it judiciously. Since the number of random numbers used in the combat process is a random variable, we use only those random numbers that we need in each column and throw away the remaining so that no overlap is possible for the next pair of runs. As a result, we maintain synchronization.

The last change we make to the crude simulation for the utilization of the antithetic variates is in the subroutine BATTLE. The subroutine BATTLE computes the values of parameters for each replication as follows:

$$B_i = (B^1{}_i + B^2{}_i) / 2 \qquad (19)$$

where

$B^1{}_i$ = the number of live bombers from the first run

$B^2{}_i$ = the number of live bombers from the second run

and

$$D_i = (D^1{}_i + D^2{}_i) / 2 \qquad (20)$$

where

$D^1{}_i$ = the number of live defenders from the first run

$D^2{}_i$ = the number of live defenders from the second run

2. Stratified Sampling

As we stated in Chapter II, stratified sampling resembles the antithetic variates procedures, and so do the changes to the crude simulation. Hence we make changes similar to those in the simulation using antithetic variates for the simulation using stratified sampling. The FORTRAN program for the BCD simulation model using stratified sampling is in Appendix D. We modify subroutines UNIFOR and EXPON, where each generates a three-dimensional array of uniform random numbers from the three strata

$$S_1 = (0,1/3), \qquad S_2 = (1/3,2/3), \qquad S_3 = (2/3,1)$$

before simulating three runs of the combat process per replication. Note this does not need to be limited to 3; we could have done more.

Gaver and Thompson (1973, p.586) describe this approach. First, a U(0,1) random number u_{11} is generated and placed in the first row and first column of the three-dimensional array. Next, 1/3 is added to the value of u_{11} with a subtraction of one if needed to get u_{12} within the range of 0 and 1. u_{12} is placed in the first row and second column of the array. Next, 1/3 is added to the value of u_{12}, and if necessary subtracted by one, to get u_{13}. u_{13} is placed in the first row of the third column of the array. If subroutine BATTLE calls for k random numbers, then k U(0,1) random numbers are generated, and the procedure obtains a value for each of the kx3 cells. BATTLE uses the first column of random numbers in the array for the first run, the second column for the second run, and the third column for the third run of the combat process.

The values of the parameters for each replication are the average values of the battle outcomes from the three runs. The subroutine BATTLE computes the values of these parameters as follows:

$$B_i = (B^1{}_i + B^2{}_i + B^3{}_i) / 3 \qquad (21)$$

where

$B^1{}_i$ = the number of live bombers from the first run

B^2_i = the number of live bombers from the second run

B^3_i = the number of live bombers from the third run

and

$$D_i = (D^1_i + D^2_i + D^3_i) / 3 \tag{22}$$

where

D^1_i = the number of live defenders from the first run

D^2_i = the number of live defenders from the second run

D^3_i = the number of live defenders from the third run

D. SUMMARY

The simulation of the BCD model is a discrete-event simulation. It begins with the initialization event and ends with termination event. The simulation of the combat process involves generating a sequence of U(0,1) random numbers to select interim event occurrences with changes in the state of the process and generating another sequence of U(0,1) random numbers to determine the unit of time until the next event occurs and to advance the simulated time of the combat process.

The programming of the antithetic variates and stratified sampling modifies crude simulation. Monotonicity and synchronization are required in generating the uniform numbers for the simulation using these VRTs. Generating random numbers by the inverse transform method guarantees monotonicity. Generating sufficient random numbers by the inverse transform method and in multi-dimensional arrays initiates and maintains synchronization.

V. APPLICATIONS OF THE ANTITHETIC VARIATES AND STRATIFIED SAMPLING

A. INTRODUCTION

We are now prepared to demonstrate the application of the variance reducing techniques to the simulation of a combat stochastic model. In this chapter we illustrate the performance of the antithetic variates (AV) and stratified sampling (SS) in the simulation of the BCD model. In Chapter IV we stated that the mean and variance of the parameters of interest estimated from simulation are used to analyze the output data of the model. Usually the estimated mean is of primary interest to decision makers, and the estimation of the variance is secondary. Since we use the variance of the parameters estimated from the simulation of the BCD model to exhibit variance reduction, we will, henceforth, focus on the variance.

We examine the applicability of AV and SS in the BCD model by simulating many scenarios of the air battle and recording increases in simulation efficiency. We investigate AV and SS performance by mapping a response surface that characterizes the efficiency of variance reduction in the model. In the next section we specify the scenarios and discuss the application of using AV and SS in the simulation of these scenarios. We then build response models that describe the performance of the AV and SS for these scenarios and discuss their experimental results in Section C. We present a brief summary of the chapter in the final section.

B. APPLICABILITY OF ANTITHETIC VARIATES AND STRATIFIED SAMPLING TO THE BCD SIMULATION MODEL

In Section B of the previous chapter we described the general scenario of the BCD model. In this section we specify various combat scenarios to observe how AV and SS are applicable to the simulation of the BCD model. Recall that seven attributes characterize a BCD scenario. Since a change of the values of one of these attribute will produce a different scenario, we chose to change the values for three attributes. We simulated 10, 30 and 50 defenders against 10, 30, and 50 bombers at "end of battle" times of 25, 75, and 125. We maintain the α rate for a defender killing a bomber at .05, the β rate for a bomber killing a defender at .05, and the ϕ rate for a bomber and a defender entering a combat engagement at .005. We also initialize every simulation with zero combat engagements. Thus, we observe the responses of the simulations at 3 "end of battle" times in 9 different scenarios. This arrangement comprises a set of 27 independent simulations.

We run this arrangement for crude simulation, simulation using AV, and simulation using SS. As a result, we perform a total of 81 different simulation experiments. In order to make a fair assessment of the applicability of AV and SS, we examine the variance obtained from the same amount of simulation or the same numbers of simulated battle runs for every simulation. We run 90 battles: this equates to 90 replications in crude simulation, 45 replications in simulation using AV, and 30 replications in simulation using SS. Table E.1 of Appendix E contains the statistical output for crude simulation; similarly, data in Tables E.2 and E.4 are from simulations using AV and SS respectively. We use Equations 6 and 7 to measure the efficiency of the variance reduction (RE) and the increase in precision of the parameters from the simulation (IP) and place the AV results in Table E.3 and SS results in Table E.5.

The values in Tables E.3 and E.5 show that AV and SS respectively are applicable in the BCD simulation model. A RE value greater than unity indicates that the VRT is effective in increasing simulation efficiency. A positive IP value exhibits their effectiveness to increase precision of the desired parameter. With these two values, we may also find the tradeoff of saving simulation time or gaining precision. We acknowledge that these values are all values of, or realizations of, random variables, but we believe that the tables show that the variance reduction adheres to a stochastic pattern. That is, the random variables obtained under certain scenarios will tend to have the same relationship to the random variables obtained under other scenarios. For example, the data in the tables suggest that high RE and IP values correspond to the scenarios that start combat with same numbers of bombers and defenders. The RE and IP values obtained under these scenarios appear consistently higher than the values under all other scenarios. Hence, the variance reduction measured by these RE and IP values are stochastically greater than the variance reduction obtained from any other scenario. Since such even combats (i.e. equal combat power) are inherently more variable in outcome, the fact that variance reduction is greatest there is certainly welcome. In the next section we attempt to conduct a more thorough investigation of these phenomena so that we may understand how the AV and SS perform in the simulation of the BCD model.

C. PERFORMANCE OF ANTITHETIC VARIATES AND STRATIFIED SAMPLING IN THE BCD SIMULATION MODEL

In the previous section we saw that AV and SS are applicable to the simulation of the BCD model. In this section we examine the variability of uncertainty in the model and then evaluate the applicability of AV and SS to reduce this uncertainty. We explore the changes in the AV and SS performance and examine the relationships of factors that affect these changes. Results of this analysis reveal the characterization of the AV and SS performance in the BCD model.

1. Experimental Design

We use the data we generated in Appendix E to fit response surfaces that describe the uncertainty in the values of parameters in the BCD model and characterize the performance of AV and SS over the prescribed range of values in the three factors: initial numbers of Bombers and Defenders and "end of battle" Time. We code the three factors as

$$x_1 = (\text{Time} - 75) / 50,$$
$$x_2 = (\text{Defender} - 30) / 20,$$
$$x_3 = (\text{Bomber} - 30) / 20.$$

Each factor has 3 levels. Thus, we may use a 3^3

factorial design to fit the data with the response surface equation

$$\begin{aligned} E(y) = {} & \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{11} x_1{*}{*}2 \\ & + \beta_{22} x_2{*}{*}2 + \beta_{33} x_3{*}{*}2 + \beta_{12} x_1{*}x_2 \\ & + \beta_{13} x_1{*}x_3 + \beta_{23} x_2{*}x_3 \end{aligned} \tag{20}$$

where

$E(y)$= expected response

β_0= intercept

β_i= linear coefficient for factor i

β_{ii}= quadratic coefficient for factor i

β_{ij}= interaction coefficient for the interaction of factors i and j

x_i= level of factor i.

We seek to obtain the maximum information from every observation; therefore, we chose a 3^3 Fractional factorial design. This design is the cuboctahedron plus three center points (John, 1971). The three center points provide an unbiased estimate of error and repeated observations which permit us to test for Lack of Fit of the response surface equation we obtained. We use the cuboctahedron design to fit data for three response surfaces: (1) variability of uncertainty inherent in the battle outcomes, (2) efficiency of AV, and (3) efficiency of SS. This design, with its three center points, and the data to which it is used to fit are shown in Tables 5.1, 5.2, and 5.3. The variances data in Table 5.1 is the variability of uncertainty inherent in the battle outcome. We obtain this data from the variance of the estimate tabulated in the crude simulation table in Appendix E. The RE values data in Table 5.2 is the efficiency of AV for the estimation of the defender and bomber parameters. We took this data from the RE values in Table E.3. In Table 5.3 is the RE values data for the efficiency of SS. This data is obtained from the RE values in Table E.5.

TABLE 5.1 DESIGN AND VARIANCES FOR A 3 x 3 EXPERIMENT ON THE VARIABILITY OF UNCERTAINTY INHERENT IN THE BATTLE OUTCOME

DECODED LEVELS			CODED LEVELS			VARIANCES	
Time	Defender	Bomber	x1	x2	x3	Defender	Bomber
25	10	30	-1	-1	0	.0328	.0541
125	10	30	1	-1	0	.0036	.1725
25	50	30	-1	1	0	.1687	.0826
125	50	30	1	1	0	.3339	.0359
25	30	10	-1	0	-1	.0458	.0350
125	30	10	1	0	-1	.2247	.0045
25	30	50	-1	0	1	.0628	.1104
125	30	50	1	0	1	.0362	.3479
75	10	10	0	-1	-1	.0387	.0358
75	50	10	0	1	-1	.1427	.0063
75	10	50	0	-1	1	.0064	.1654
75	50	50	0	1	1	.2717	.2865
75	30	30	0	0	0	.1618	.1895
75	30	30	0	0	0	.0925	.0966
75	30	30	0	0	0	.1693	.1589

TABLE 5.2 DESIGN AND RE VALUES FOR A 3 x 3 EXPERIMENT ON THE EFFICIENCY OF THE ANTITHETIC VARIATES IN THE BCD MODEL

DECODED LEVELS			CODED LEVELS			RE VALUES	
Time	Defender	Bomber	x1	x2	x3	Defender	Bomber
25	10	30	-1	-1	0	2.5	5.4
125	10	30	1	-1	0	1.5	3.9
25	50	30	-1	1	0	6.0	2.2
125	50	30	1	1	0	7.2	1.4
25	30	10	-1	0	-1	3.9	3.0
125	30	10	1	0	-1	4.0	1.5
25	30	50	-1	0	1	2.8	3.6
125	30	50	1	0	1	1.8	10.9
75	10	10	0	-1	-1	8.1	6.0
75	50	10	0	1	-1	3.5	1.1
75	10	50	0	-1	1	1.1	4.2
75	50	50	0	1	1	10.1	9.3
75	30	30	0	0	0	6.4	7.0
75	30	30	0	0	0	13.5	7.0
75	30	30	0	0	0	10.8	9.2

TABLE 5.3 DESIGN AND RE VALUES FOR A 3 x 3 EXPERIMENT ON THE EFFICIENCY OF STRATIFIED SAMPLING IN THE BCD MODEL

DECODED LEVELS			CODED LEVELS			RE VALUES	
Time	Defender	Bomber	x1	x2	x3	Defender	Bomber
25	10	30	-1	-1	0	2.6	2.1
125	10	30	1	-1	0	.9	2.2
25	50	30	-1	1	0	3.0	1.9
125	50	30	1	1	0	2.2	1.2
25	30	10	-1	0	-1	2.0	2.8
125	30	10	1	0	-1	2.6	1.7
25	30	50	-1	0	1	2.9	1.4
125	30	50	1	0	1	1.7	2.6
75	10	10	0	-1	-1	1.6	1.5
75	50	10	0	1	-1	1.6	1.2
75	10	50	0	-1	1	2.0	1.5
75	50	50	0	1	1	3.0	3.3
75	30	30	0	0	0	3.7	2.7
75	30	30	0	0	0	2.9	3.0
75	30	30	0	0	0	2.5	1.5

2. Experimental Analysis

To perform the statistical analysis, we use the Response Surface Regression (RSREG) procedure in the Statistical Analysis System (SAS) computer software package on the IBM 370 mainframe. With this procedure we were able to obtain a second order response-surface equation by least-square regression, check for model adequacy, test for lack-of-fit, and identify critical surface values which were useful in helping to describe the shape of the surface.

We fitted Equation 20 to the data in the respective design tables in the previous section and obtained multiple response surface equations and multiple analysis of variance (ANOVA) tables for corresponding responses. We assess the adequacy of each equation and test for fit from its corresponding ANOVA table. From each response surface equation, we generated additional data to obtain contour plots. We plotted contours of variability of uncertainty for the initial numbers of bombers and defenders at various Times. Similarly, we plotted contours of the efficiencies of AV and SS for initial numbers of bombers and defenders. From these plot we were able to see how the initial numbers of bombers and defenders in combat affect the variability of uncertainty in the battle outcomes and the AV and SS performance.

a. Variability of uncertainty of the battle outcomes

Equation 21 provides an adequate description of the response surface that characterizes the variability of uncertain in the expected numbers of live defenders at the end of the battle in the BCD model. The response $V_{Defender}$ is the expected amount of uncertainty in the defender estimate.

$$
\begin{aligned}
V_{Defender} = {} & .1412 + .036x_1 + .104x_2 - .009x_3 - .014x_1{**}2 \\
& + .049x_1x_2 + .008x_2{**}2 - .051x_1x_3 \\
& + .040x_2x_3 - .034x_3{**}2 \qquad (21)
\end{aligned}
$$

TABLE 5.4 ANOVA FOR THE EXPECTED AMOUNT OF UNCERTAINTY IN THE DEFENDER ESTIMATE

SOURCE	d.f.	SS	MS	F-RATIO
Fitted Surface	9	.1302	.0145	8.06
Lack of Fit	3	.0102	.0034	1.90
Pure Error	2	.0036	.0018	
Total	14	.1440		

R-Square=.9043   Mean Variance=.1194   Std. Dev.=.0525

We see from ANOVA Table 5.4 that the variation in the variance of live defenders is insignificant at the 95% level (its F-Ratio of 8.06 is less than $F_{.95;9,2}$ = 19.38). The Lack of fit is also insignificant (F-Ratio= 1.90 < $F_{.95;3,2}$ = 19.16). The R-Square value is .9043 which indicates that about 90% of the variation in the variance of the expected number of live defenders is accounted for by Equation 21. Further analysis reveals that the response surface is shaped like a rising ridge. The plots of contours at Figure 5.1 illustrate the nature of this Response surface. These pictures show that initial numbers of bombers and defenders affect the variance of defenders at Time 25. At Times 75 and 125 the initial numbers of bombers have little influence. Here the variance of the expected number of live defenders is affected solely by the increase in the number of initial defenders. Hence, as the number of defenders increases, the variability of uncertainty in the estimate of the expected number of live defenders at the end of the battle increases.

Equation 22 provides an adequate description of the response surface that characterizes the variability of uncertainty in the expected numbers of live bombers at the end of the battle in the BCD model. V_{Bomber} is the expected amount of uncertainty in the Bomber estimate.

$$\begin{aligned} V_{Bomber} = {} & .1483 + .035x_1 - .002x_2 + .104x_3 - .031x_1{**}2 \\ & - .041x_1x_2 + .032x_2{**}2 - .067x_1x_3 \\ & + .038x_2x_3 - .007x_3{**}2 \qquad (22) \end{aligned}$$

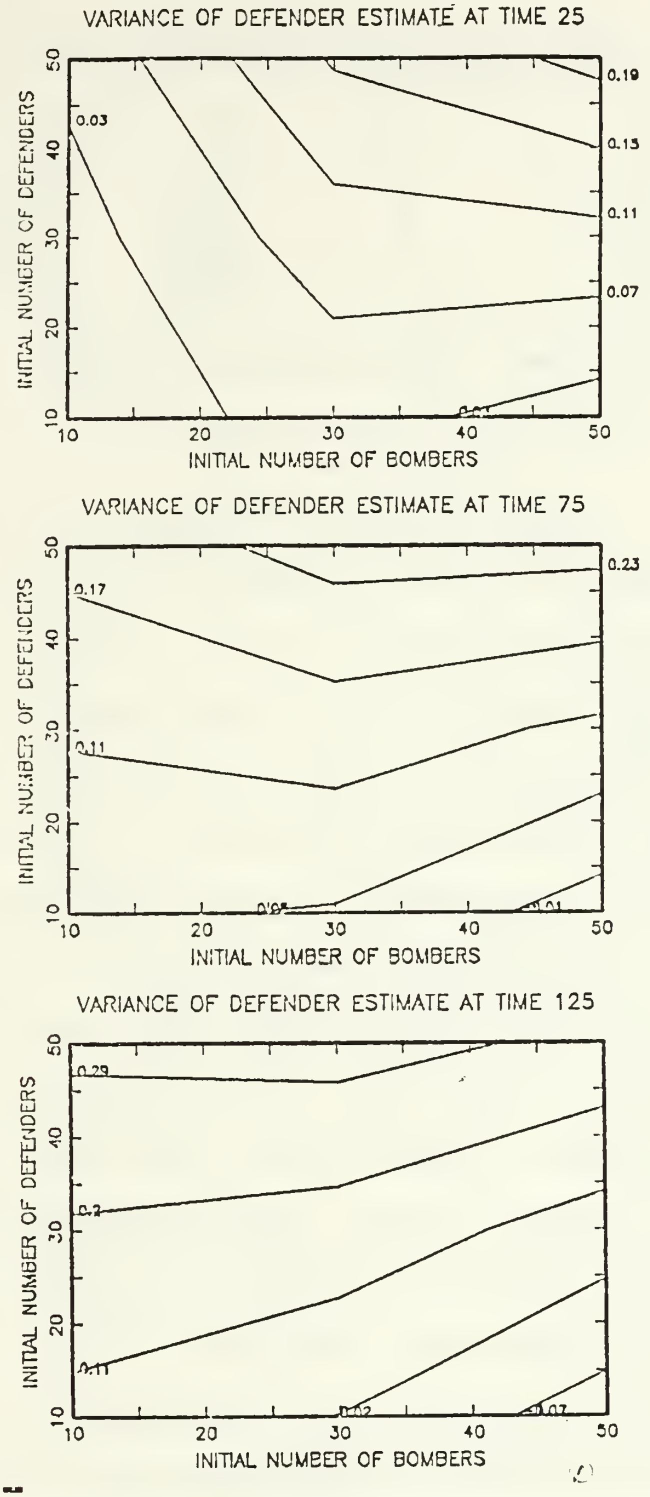


Figure 5.1 Contour Plots of the Response Surface for the Expected Amount of Uncertainty in the Defender Estimate

TABLE 5.5 ANOVA FOR THE EXPECTED AMOUNT OF UNCERTAINTY IN THE BOMBER ESTIMATE

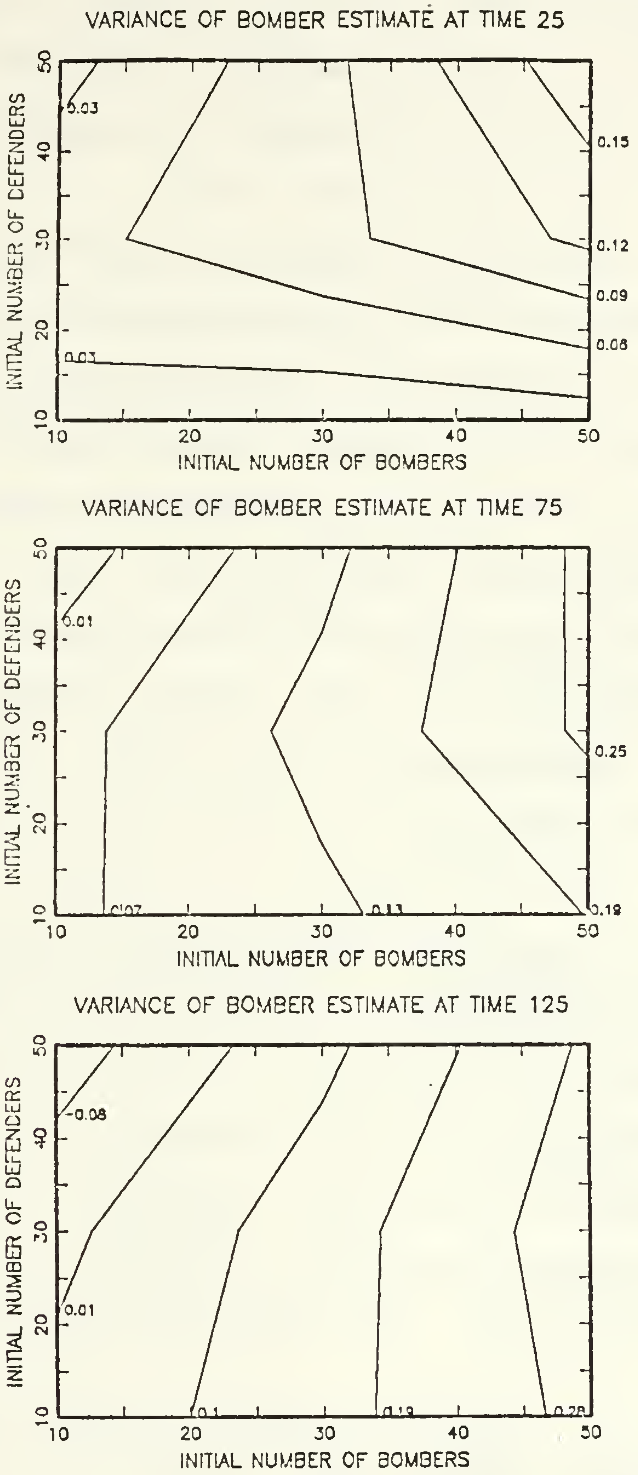
SOURCE	d.f.	SS	MS	F-RATIO
Fitted Surface	9	.1331	.0149	6.77
Lack of Fit	3	.0073	.0024	1.08
Pure Error	2	.0045	.0022	
Total	14	.1408		

R-Square=.9188 Mean Variance=.1188 Std. Dev.=.0485

ANOVA Table 5.5 shows that the variation in the variance of live bombers is insignificant at the 95% level (its F-Ratio of 6.77 is less than $F_{.95;9,2} = 19.38$). The Lack of fit is also insignificant (F-Ratio= $1.08 < F_{.95;3,2} = 19.16$). The R-Square value indicates that about 92% of the variation in the variance of live bombers is accounted for by Equation 22. Further analysis reveals that this response surface is also shaped like a rising ridge. The plots of contours at Figure 5.2 illustrate the nature of this response surface. These figures show that the variance of the expected number of live bombers generally increase as the initial number of bombers increases.

b. Antithetic Variates.

Equation 23 provides an adequate description of an AV response surface in the estimation of the expected numbers of live defenders.



VARIANCE OF BOMBER ESTIMATE AT TIME 25

VARIANCE OF BOMBER ESTIMATE AT TIME 75

VARIANCE OF BOMBER ESTIMATE AT TIME 125

Figure 5.2 Contour Plots of the Response Surface for the Expected Amount of Uncertainty in the Bomber Estimate

$$A_{Defender} = 10.20 - .09x_1 + 1.70x_2 - .46x_3 - 4.25x_1{**}2 + .55x_1x_2 - 1.68x_2{**}2 - .28x_1x_3 + 3.40x_2x_3 - 2.85x_3{**}2 \quad (23)$$

We see from ANOVA Table 5.6 that the variation in the efficiency of AV is insignificant at the 95% level (its F-Ratio of 1.46 is less than $F_{.95;9,2} = 19.38$). The Lack of fit is also insignificant (F-Ratio=.46 < $F_{.95;3,2} = 19.16$). The R-Square value is .8497 which indicates that about 85% of the variation in the Defender RE values is accounted for by Equation 23. Here, $A_{Defender}$ is the expected simulation efficiency of AV generated to reduce the uncertainty in the Defender estimate.

TABLE 5.6 ANOVA FOR THE EXPECTED EFFICIENCY OF AV (RE VALUE) GENERATED TO REDUCE THE UNCERTAINTY IN THE DEFENDER ESTIMATE

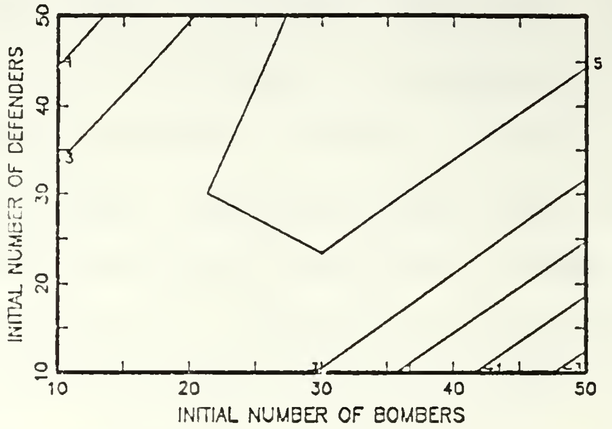
SOURCE	d.f.	SS	MS	F-RATIO
Fitted Surface	9	168.31	18.70	1.46
Lack of Fit	3	4.08	1.36	.47
Pure Error	2	25.69	12.84	
Total	14	198.08		

R-Square=.8497 Mean RE Value=5.54 Std. Dev.=2.44

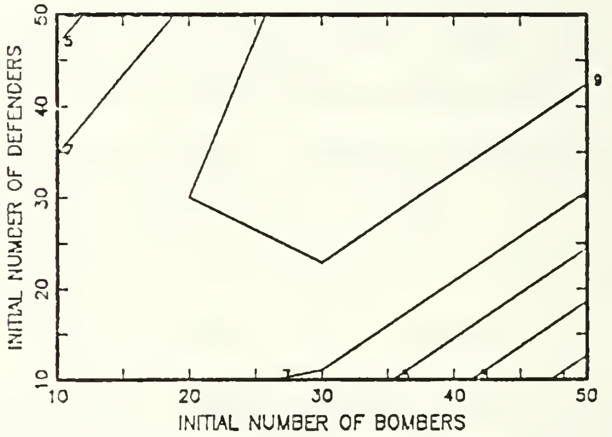
Further analysis reveals that the response surface is shaped like a hill with a gentle slope on one side and a fairly steep slope on the other side. The maximum value of this surface occurs in the BCD scenario that begins combat with 50 defenders and 40 bombers and ends the battle at time 75. The plots of contours at Figure 5.3 illustrate the nature of this AV response surface. These pictures clearly show how AV performs in different scenarios for times of 25, 75, and 125. Beside having its best performance in a scenario that ends at time 75, AV appears to be strong in scenarios that initiate the air battle with at least 30 defenders and 30 bombers. Its weakest performance seems to occur in those scenarios that commence combat with no more than 30 defenders and 30 bombers. The plots of contours show that the efficiency of AV is subversive in those scenarios whose simulation initializes the air battle with 30 or less defenders and 40 or more bombers. For these scenarios, simulation efficiency of AV may often increase, instead of decrease, the uncertainty in the Defender estimate. We will discuss why this is so in the Experimental Result Section. Similar analysis of the AV performance is made for the Bomber RE values. An adequate description of the AV response surface in the estimation of the bombers is characterized by

$$\begin{aligned} Y_{Bomber} = {} & 7.73 + .44x_1 - .69x_2 + 2.05x_3 - 2.45x_1{**}2 \\ & + .18x_1x_2 - 2.05x_2{**}2 + 2.20x_1x_3 \\ & + 2.50x_2x_3 - .53x_3{**}2 \qquad (24) \end{aligned}$$

EFFICIENCY OF AV IN DEFENDER ESTIMATE AT TIME 25



EFFICIENCY OF AV IN DEFENDER ESTIMATE AT TIME 75



EFFICIENCY OF AV IN DEFENDER ESTIMATE AT TIME 125

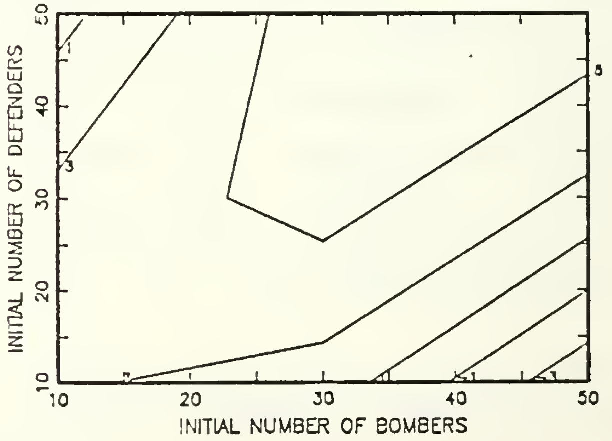


Figure 5.3 Contours Plots of the Responses Surface for the Expected Efficiency of AV Generated to Reduce the Uncertainty in the Defender Estimate

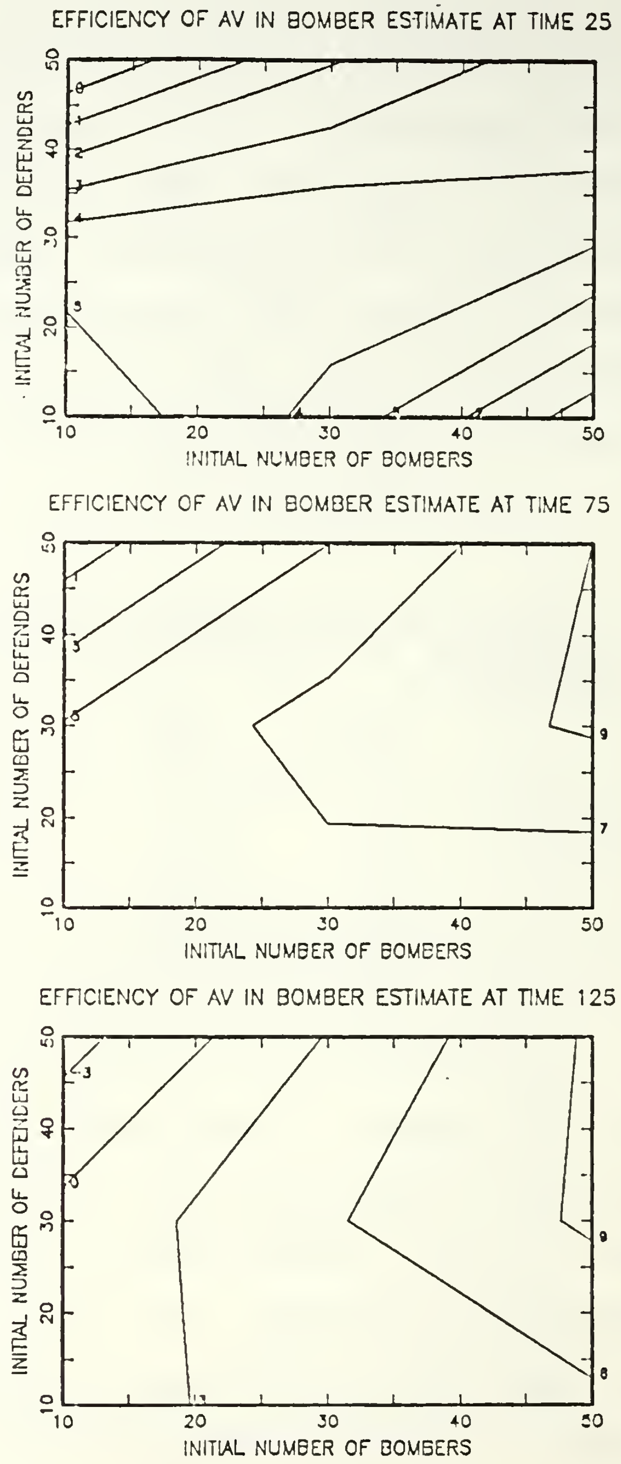
Y_{Bomber} is the expected simulation efficiency of AV to reduce the uncertainty in the Bomber estimate. The ANOVA Table 5.7 indicates that the proportion of the total variation in the Bomber RE values accounted for in Equation 24 is over 87%. Furthermore, this variation is insignificant at the 95% level (F-ratio value of 8.19); lack of fit is also insignificant (F-Ratio= 1.61).

TABLE 5.7 ANOVA FOR THE EXPECTED EFFICIENCY OF AV (RE VALUE) GENERATED TO REDUCE THE UNCERTAINTY IN THE BOMBER ESTIMATE

SOURCE	d.f.	SS	MS	F-RATIO
Fitted Surface	9	118.74	13.19	8.19
Lack of Fit	3	14.17	4.72	1.61
Pure Error	2	3.23	1.61	
Total	14	136.14		

R-Square=.8722 Mean RE Value=5.05 Std. Dev.=1.87

Examining this response surface further we find that the shape of the surface changes over time. The plots of contours depicted in Figure 5.4 show that the shape of the surface looks like a saddle at Time 25, a gentle slope at Time 75, and a uniformly rising ridge at Time 125. The critical values for this surface also change as its shape changes. Most notable are the values for maximum efficiency. At Time 25, maximum



EFFICIENCY OF AV IN BOMBER ESTIMATE AT TIME 25

EFFICIENCY OF AV IN BOMBER ESTIMATE AT TIME 75

EFFICIENCY OF AV IN BOMBER ESTIMATE AT TIME 125

Figure 5.4 Contour Plots of the Response Surface for the Expected Efficiency of AV Generated to Reduce the Uncertainty in the Bomber Estimate

efficiency occurs in scenarios that begin fighting with less than 20 defenders and bomber. By Time 125, maximum efficiency has shifted to the scenarios that start with at least 40 defenders and bombers. The least amount of AV reduction occurs, at any Time, in those scenarios that initialize the combat simulation with more than 30 defenders and less than 10 bombers.

Here is a summary of what is revealed by the above analysis. AV, in general, seems to be the strongest and most consistent, and equally-distributed between closely-matched pairs of bombers and defenders. Furthermore, the larger the evenly-matched contest the greater the variance reduction. When the defenders and bombers are not evenly matched, AV is not as consistent and does not provide equal variance reduction in the estimation of the pair of parameters. It is strong in the estimation of the larger combatants and weak in the estimation of the smaller ones.

c. Stratified Sampling.

We analyze the efficiency of SS in the simulation of the BCD model in a similar manner as we analyzed the efficiency of AV. If we analyze Equations 25 and 26 in terms of ANOVA Tables 5.8 and 5.9 respectively, we will get results similar to those we obtained in the last section. Therefore, we will forego this particular analysis. Note that $Y_{Defender}$ is the expected simulation efficiency of SS to reduce the uncertainty in the Defender estimate, and Y_{Bomber} is the expected simulation efficiency of SS to decrease the uncertainty in the Bomber estimate.

$$Y_{Defender} = 3.03 - .39x_1 + .49x_2 + .08x_3 - .15x_1{**}2 + .22x_1x_2 - .70x_2{**}2 - .45x_1x_3 + .55x_2x_3 - .58x_3{**}2 \quad (25)$$

TABLE 5.8 ANOVA FOR THE EXPECTED EFFICIENCY OF SS (RE VALUE) GENERATED TO REDUCE THE UNCERTAINTY IN THE DEFENDER ESTIMATE

SOURCE	d.f.	SS	MS	F-RATIO
Fitted Surface	9	8.24	.92	2.49
Lack of Fit	3	.53	.18	.47
Pure Error	2	.75	.37	
Total	14	9.52		

R-Square=.86661 Mean RE Value=2.27 Std. Dev.=.51

$$Y_{Bomber} = 2.90 - .06x_1 - .03x_2 + .21x_3 - .41x_1{**}2 - .20x_1x_2 - .64x_2{**}2 + .58x_1x_3 + .50x_2x_3 - .36x_3{**}2 \quad (26)$$

TABLE 5.9 ANOVA FOR THE EXPECTED EFFICIENCY OF SS (RE VALUE) GENERATED TO REDUCE THE UNCERTAINTY IN THE BOMBER ESTIMATE

SOURCE	d.f.	SS	MS	F-RATIO
Fitted Surface	9	5.18	.58	19.33
Lack of Fit	3	1.82	.61	20.33
Pure Error	2	.06	.03	
Total	14	7.06		

R-Square=.7340 Mean RE Value=2.15 Std. Dev.=.61

The contour plots at Figures 5.5 and 5.6 appear to have similar features. They show a relatively flat surface except at the corners. The corner with 50 Bombers and 50 Defenders has the highest response and the other corners have low response. These plots suggest that the SS performance is generally consistent in all but a few scenarios in the BCD model. Maximum efficiency of SS occurs in those scenarios that initialize simulation with equally large numbers of bombers and defenders. It is very weak in those scenarios that begin combat with either less than 10 defenders and more than 40 bombers or more than 40 defenders and less than 10 bombers.

3. Experimental Results

The experimental results can be summarized in Tables 5.10 and 5.11. Table 5.10 shows the relationships between the AV performance and the uncertainty in the Defender estimate and the SS

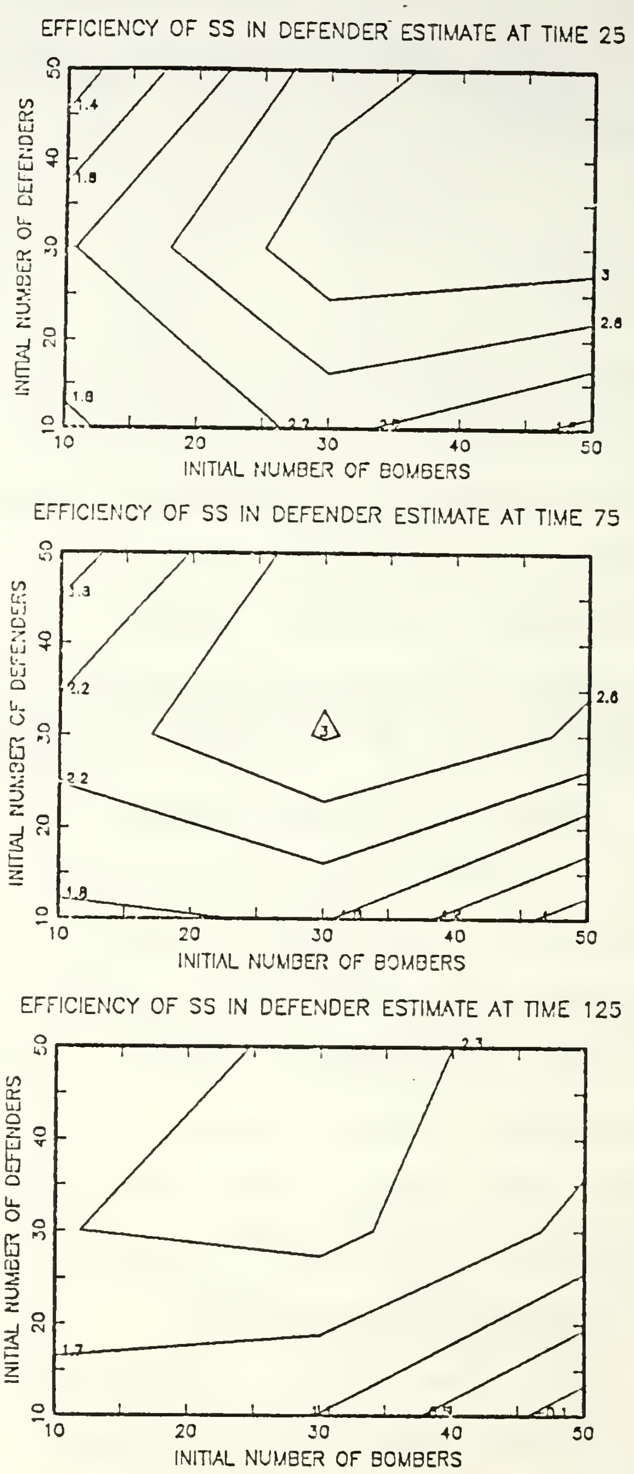
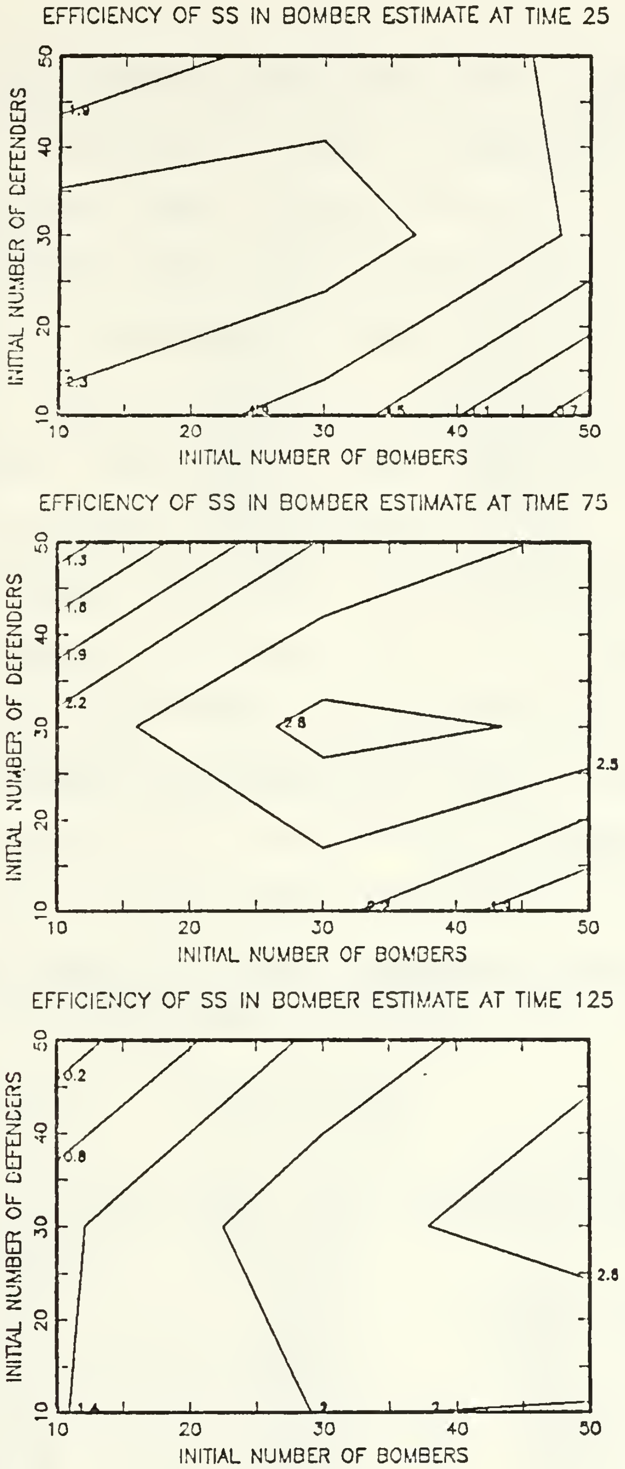


Figure 5.5 Contour Plots of the Response Surface for the Expected Efficiency of SS Generated to Reduce the Uncertainty in the Defender estimate


EFFICIENCY OF SS IN BOMBER ESTIMATE AT TIME 25

EFFICIENCY OF SS IN BOMBER ESTIMATE AT TIME 75

EFFICIENCY OF SS IN BOMBER ESTIMATE AT TIME 125

Figure 5.6 Contour Plots of the Response Surface for the Expected Efficiency of SS Generated to Reduce the Uncertainty in the Bomber Estimate

performance and the uncertainty in the Defender estimate. Similarly, Table 5.11 shows the relationships between the AV performance and the uncertainty in the Bomber estimate and the SS performance and the uncertainty in the Bomber estimate.

We further examine this relationship by analyzing the data that measure the uncertainty (crude variance) and appropriate variance reduction (RE Values) in Appendix E. After we applied a logarithmic transformation to the data, we regress the RE values on the crude variance data and observe a strong logarithmic linear relationship between uncertainty and variance reduction.

TABLE 5.10 RELATIONSHIP BETWEEN UNCERTAINTY AND THE EFFICIENCY OF VARIANCE REDUCTION IN THE DEFENDER ESTIMATE

INITIAL		UNCERTAINTY	VARIANCE REDUCTION	
Defenders	Bombers	Variance	AV	SS
10	10	medium	strong	fair
10	30	medium	fair	fair
10	50	small	weak	weak
30	10	large	strong	fair
30	30	large	strong	strong
30	50	medium	fair	fair
50	10	large	strong	weak
50	30	large	strong	fair
50	50	large	strong	fair

TABLE 5.11 RELATIONSHIP BETWEEN UNCERTAINTY AND THE EFFICIENCY OF VARIANCE REDUCTION IN THE BOMBER ESTIMATE

INITIAL		UNCERTAINTY	VARIANCE REDUCTION	
Defenders	Bombers	Variance	AV	SS
10	10	medium	strong	fair
10	30	large	strong	fair
10	50	large	strong	fair
30	10	small	fair	fair
30	30	large	strong	fair
30	50	large	strong	fair
50	10	small	weak	weak
50	30	medium	fair	fair
50	50	large	strong	fair

This relationship is manifested in the multiplicative equation shown in Table 5.12. VAR is the value of uncertainty or variance of the corresponding estimate obtained from crude simulation, and Y is the simulation efficiency of the variance reduction or RE value for the corresponding estimate.

TABLE 5.12 ANALYSIS OF THE RELATIONSHIP BETWEEN UNCERTAINTY AND EFFICIENCY OF VARIANCE REDUCTION IN PARAMETER ESTIMATES

VRT	ESTIMATE	RELATIONSHIP	CORRELATION COEFFICIENT	SET POINT
AV	Defender	Y= 10.43 * VAR**.362	.8609	.00154
AV	Bomber	Y= 9.73 * VAR**.362	.8217	.00186
SS	Defender	Y= 3.18 * VAR**.144	.5601	.00032
SS	Bomber	Y= 2.99 * VAR**.147	.6054	.00058

The correlation coefficient reveals the strength of the logarithmic linear relationships between uncertainty and the efficiencies of AV and SS. With the values of the exponent in the equations being less than one and the values of the correlation coefficient being positive, the efficiencies of AV and SS are observed to increase, at a decreasing rate, as the uncertainty (variance) increases. We obtained the Set Point by setting Y=1 in the corresponding equation and solving for VAR. At this value, simulation efficiency nether increases nor decreases. Now if we observe a value of uncertainty, or variance obtained from crude simulation, above this set point, then we expected to get an efficiency of a VRT to increase the simulation efficiency. On the other hand, if the value is below the set point, then we expect the efficiency of the VRT to decrease the simulation efficiency.

Here is the bottom line on AV and SS performance in the BCD model:

1. If we apply antithetic variates to the simulation of the BCD model,
    a. We may expect the variability of uncertainty in the defender estimate
        (1) to decrease if the variance of the estimate obtained from crude simulation is at least .00154, and

        (2) to decrease, at a decreasing rate, with an increase in the variance of the estimate obtained from the crude simulation.

    b. We may expect the variability of uncertainty in the bomber estimate

        (1) to decrease if the variance of the estimate obtained from the crude simulation is at least .00186, and

        (2) to decrease, at a decreasing rate, with an increase in the variance of the estimate obtained from the crude simulation.

2. If we apply stratified sampling to the simulation of the BCD model,

    a. We may expect the variability of uncertainty in the defender estimate

        (1) to decrease if the variance of the estimate obtained from crude simulation is at least .00032, and

        (2) to decrease, at a decreasing rate, with an increase in the variance of the estimate obtained from the crude simulation.

    b. We may expect the variability of uncertainty in the bomber estimate

        (1) to decrease if the variance of the estimate obtained from the crude simulation is at least .00058, and

        (2) to decrease, at a decreasing rate, with an increase in the variance of the estimate obtained from the crude simulation.

E. SUMMARY

We illustrated the performance of the antithetic variates and stratified sampling in the simulation of the BCD model. The manifestation of their pair performance was characterized by response surface equations and plots of contour lines. Both VRTs were shown to be effective in increasing simulation efficiency, but they perform differently in the BCD model. AV provides the largest amount of variance reduction but is more volatile. AV increases the simulation efficiency on the average of 5 times the crude simulation; it is strong in the BCD scenarios where there is large amount of uncertainty in the battle outcomes for live bombers and defenders, and weak in those scenarios where there is little amount of uncertainty in the battle outcomes. SS, on the hand, has a more consistent performance. SS increases the simulation efficiency at a mean of 2 times the crude simulation. It performs nearly the same in every scenario except where the uncertainty is large.

VI. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

The objective of this pilot study has been to investigate the applicability of VRTs to reduce the inherent variability in stochastic combat models. We examined the effects of applying AV and SS to the simulation of a simple stochastic combat model. We have now shown that AV and SS are applicable to this stochastic combat model. We can infer that these VRTs are indeed capable of working in stochastic combat models, and their prospects in larger and more complex stochastic combat models are even more promising. The conditions of monotonicity and synchronization are essential parts of the design of the simulation program for these models. Hence, we feel that sizable increase in simulation efficiency is possible if these requirements are met in the simulation.

The experimental results of applying VRTs to the BCD simulation model show that the strength of AV and SS is influenced by uncertainty. A strong variance reduction results from a large variance of the estimate obtained from crude simulation. A weak variance reduction is caused from a small variance of the estimate obtained from crude simulation. Hence, sizable and consistent variance reduction depends on large variability of the simulated output from the stochastic combat model. Therefore, the variability of the output from larger and more complex stochastic models must also be large enough to obtain the size and consistency of simulation efficiency and variance reduction one desires from the applications of these VRTs to such models.

B. RECOMMENDATIONS

The pilot study presented in this thesis provides a base for further studies in the applications of AV and SS to large-scale, real world, stochastic combat simulation models. Usually complex simulation models have many subroutines or modules. The variability of uncertainty in the output data from these modules may vary from low to high. We recommend that a study of this matter focus on the degree of variability of uncertainty in the output data from each module. The interest of the study should be concerned with the relationship between the performance of the VRT and the variability of the output data from each module. The results should indicate where and how the VRT may be used in the model in order to maximize the simulation efficiency of the model. For example, if the study shows that a VRT performs strongly in a particular module whose variability of output data is large and it performs poorly in another module whose variability of output is small, then the study should recommend that the VRT be used in the module in which it performs best and not be used in that module for which it performs poorly. Using VRTs in the module with small variability would most likely decrease simulation efficiency for that model and, at worst, suboptimize the overall performance of the VRT in the complex combat model.

APPENDIX A

FORMULATION OF THE BCD MARKOVIAN MODEL

by

Professor Donald P. Gaver
Naval Postgraduate School
Monterey, California

B bombers are approaching a group of D defenders. When the two groups approach within range each defender searches for a bomber; after he finds one they engage in combat. Either bomber or defender may win the combat; the survivor becomes "free", and is a candidate for the next combat. In general, bombers attempt to avoid combat, defenders seek it out.

This situation becomes a tri-variate Markov chain if the following state is defined: $\{B(t),C(t),D(t)\}$. Here t is conveniently measured from the time bombers and defenders are close enough to permit combat at all, $B(t)$ is the number of free bombers at time t thereafter; ditto for $D(t)$, the number of free defenders; $C(t)$ is the number of one-on-one combats. Here are a set of transition rates:

(1) Combats begin. If $B(t)=b$, $C(t)=c$, $D(t)=d$ and if Φ is the rate at which free bombers are found by free defenders then with probability

$$((e^{-\phi t})^b)^d = 1 - \phi bdt + o(t)$$

no defender finds a bomber in time t. Hence the probability that a defender does find a bomber is $\phi bdt + o(t)$. This is the rate at which free bombers and defenders get converted to combats: the state

$$(b, c, d) \rightarrow (b-1, c+1, d-1) \text{ with prob } \phi bdt.$$

(2) Defenders win. Same initial conditions. If a bomber is in combat with a defender the probability that a defender shoots down the bomber in time t (combat duration) if the latter doesn't hit the defender is

$$P\{\text{Combat duration} \leq t \mid \text{bomber doesn't hit}\}$$
$$= 1 - e^{-\alpha t}.$$

Likewise, the probability that the bomber shoots down the defender is

$$P\{\text{Combat duration} \leq t \mid \text{defender doesn't shoot}\}$$
$$= 1 - e^{-\beta t}.$$

Now suppose they both shoot, doing so independently. Model the probability that both survive to t as

$$P\{\text{Combat duration} > t\} = (e^{-\alpha t})(e^{-\beta t}).$$

Now the probability that combat lasts until t and is terminated by a defender shooting down the bomber is

$$P\{\text{Combat ends is } (dt), \text{ Defender wins}\}$$
$$= (e^{-\alpha t}\alpha dt)e^{-\beta t}$$
$$= (e^{-(\alpha + \beta)t}(\alpha + \beta))(\alpha/(\alpha+\beta).$$

This shows that a single combat duration is $\exp(\alpha + \beta)$ and the event of a defender's winning is independently $\alpha/(\alpha + \beta)$. Likewise, the combat duration is $\exp(\alpha + \beta)$ and a bomber's win is, independently, $\beta/(+ \beta)$. If there are c combats going on then the first combat to ends does so in time $\exp(c(\alpha + \beta))$.

Hence

$(b, c, d) \to (b, c-1, d+1)$ with prob αcdt

$(b, c, d) \to (b+1, c-1, d)$ with prob cdt.

(3) Simulation and Sojourn. The above shows that we may simulate the combat as follows.

(i) You are in state (b, c, d). Obtain a sojourn time in that state that is

$$\exp(\Phi bd + (\alpha + \beta)c)$$

i.e. $S_{bcd} = 1 / (\Phi bd + (\alpha + \beta)c)$.

The system stays in state (b,c,d) for time $[0, S_{bcd})$.

(ii) With probability

$$\Phi bd / (\Phi bd + (\alpha + \beta)c) \rightarrow (b-1,\ c+1,\ d-1)$$

(NEW COMBAT) at time S_{bcd}.

(iii) With probability

$$\alpha c / (\Phi bd + (\alpha + \beta)c) \rightarrow (b,\ c-1,\ d+1)$$

(DEFENDER SHOOTS DOWN BOMBER) at time S_{bcd}.

(iv) With probability

$$\beta c / (\Phi bd + (\alpha + \beta)c) \rightarrow (b+1, c-1, d)$$

(BOMBER SHOOTS DOWN DEFENDER) at time S_{bcd}.

APPENDIX B

FORTRAN PROGRAM LISTING FOR THE CRUDE SIMULATION OF THE BCD MODEL

```
      DIMENSION BX(101),DX(101),SEED(2),BOX(2),DOX(2)
      INTEGER I,N,BB,DD,R
      REAL*4 X,Y,Z,BX,DX,TXT,BOX,DOX
      DOUBLE PRECISION SEED
      DATA SEED /1234.0D0,567890123.0D0/
      R= 2
      N=100
C
C  RECEIVE INPUT DATA FROM TERMINAL
C
      WRITE(*,3)
 3    FORMAT(1X,'ENTER THE NUMBER OF BOMBERS')
      READ '(I2)', BB
      WRITE(*,4)
 4    FORMAT(1X,'ENTER THE RATE WHICH A  BOMBER SHOOTS
      DOWN A DEFENDER')
      READ '(F5.3)', Y
      WRITE(*,5)
 5    FORMAT(1X,'ENTER THE NUMBER OF DEFENDERS')
      READ '(I2)', DD
      WRITE(*,6)
 6    FORMAT(1X,'ENTER THE RATE WHICH A DEFENDER'
     & 'SHOOTS DOWN A BOMBER')
      READ '(F5.3)', X
      WRITE(*,7)'ENTER THE RATE WHICH FREE DEFENDERS'
     & 'FIND FREE BOMBERS'
 7    FORMAT(1X,A)
      READ '(F5.3)', Z
      WRITE(*,8)'ENTER THE  TIME DURATION OF THE
     & 'BATTLE'
 8    FORMAT(1X,A)
      READ '(F5.3)', TXT
C
C   RUN REPLICATIONS OF N BATTLES AND OBTAIN SUMMARY OF
C   N BATTLES
C
      CALL BATTLE(N,R,SEED,X,Y,Z,BB,DD,TXT,BX,DX)
C
C  COMPUTE STATISTICAL OUTPUT ANALYSIS OBTAIN PARAMETER
c  ESTIMATES
C
      CALL STAT(N,R,BX,DX,BOX,DOX)
C
```

```
C  FORMAT AND PRINT OUTPUT OF PARAMETER ESTIMATES
C
       WRITE(3,279) N
 279   FORMAT(15X,'SAMPLE SIZE',I6,/15X,6('-'),1X,
     &  4('-'))
       WRITE(3,280) BB,DD
 280   FORMAT(//1X,I4,2X,'BOMBERS',6X,'VERSUS',6X,
     & I4,2X,'DEFENDERS',/1X,13('-'),18X,14('-'))
 290   WRITE (3,300)TXT
 300   FORMAT(//18X,'TIME',F6.1,/,47('-'))
       WRITE(3,310)
 310   FORMAT(19X,'BOMBER',2X,'DEFENDER',/18X,
     & 7('='),2X,8('='))
       WRITE(3,320) BOX(1),DOX(1)
 320   FORMAT(1X,'AVERAGE',7X,2F10.4)
       WRITE(3,330) BOX(2),DOX(2)
 330   FORMAT(1X,'VARIANCE',6X,2F10.4)
       STOP
       END
C
C
C   SUBROUTINE BATTLE
C
       SUBROUTINE
     & BATTLE(N,R,SEED,X,Y,Z,BB,DD,TXT,BX,DX)
       INTEGER BB,DD,I,K,N,R
       REAL X(N+1),DX(N+1),GA,SOJ,X,Y,Z,B,C,D,NC,
     & BW,DW,INF,T,TIME,TXT
       DOUBLE PRECISION SEED(R)
C
C  INITIALIZE STATISTICAL COUNTERS
C
       BX(N+1)= 0.0
       DX(N+1)= 0.0
C
C  RUN N REPLICATIONS
C
       DO 200 I=1,N
C
C   INITIALIZE START-TO-BATTLE VALUES
C
                T=0.0
                C=0.0
                B=REAL(BB)
                D=REAL(DD)
                BW= 0.0
                DW= 0.0
                NC= 1.0
C
C  OBSERVE OCCURRENCE OF AN EVENT
C
```

```
 100            CALL UNIFOR(SEED(1),GA,1)
C
C DETERMINE  NEXT INTERIM EVENT AND UPDATE THE  STATE
C  (B,C,D)
C
              IF(GA .LE. BW) THEN
                B=B + 1.0
                C=C - 1.0
                D=D
              ELSE IF (GA .LE. (BW+NC)) THEN
                B=B - 1.0
                C=C + 1.0
                D=D - 1.0
              ELSE
                B=B
                C=C - 1.0
                D=D + 1.0
              END IF
C
C  COMPUTE MEAN TIME IN STATE (B,C,D)
C
              IF ((B .EQ. 0.0 .OR. D .EQ. 0.0)  .AND. C
     &       .EQ. 0.0) THEN
                  INF= 1000000.0
              ELSE
                  INF= 1.0/(Z*D*B + (X + Y)*C)
              END IF
C
C  GENERATE SOJOURN TIME IN STATE (B,C,D)
C
              CALL EXPON(SEED(2),SOJ,1)
              TIME= -INF * ALOG(SOJ)
C
C  ADVANCE THE SIMULATED TIME OF THE AIR BATTLE
C
              T= T + TIME
C
C  COMPUTE PROBABILITY OF NEXT INTERIM EVENTS OCCURRING

C
                BW= Y*C*INF
                DW= X*C*INF
                NC= Z*B*D*INF
C
C  CHECK CONDITIONS FOR OCCURRENCE OF TERMINATION EVENT

C
              IF (T .LT. TXT) GOTO 100
C
C   ACCUMULATE SUMMARY OF N BATTLE OUTCOMES
C
```

```
                BX(I)= B + C
                DX(I)= D + C
                BX(N+1)= BX(N+1) + BX(I)
                DX(N+1)= DX(N+1) + DX(I)
 200      CONTINUE
          RETURN
          END
C
C
C    SUBROUTINE EXPON
C
          SUBROUTINE EXPON(SEED2,A2,K)
          INTEGER I,K
          REAL STIM,A2,INF
          DOUBLE PRECISION EFF,SEED2
          EFF= 2147483647.0D0
          SEED2=DMOD(16807.0D0 * SEED2,EFF)
          A2= SEED2/EFF
          RETURN
          END
C
C
C    SUBROUTINE UNIFOR
C
          SUBROUTINE UNIFOR(SEED1,A1,K)
          INTEGER I,K
          REAL A1
          DOUBLE PRECISION EFF,SEED1
          EFF= 2147483647.0D0
          SEED1=DMOD(16807.0D0 * SEED1,EFF)
          A1= SEED1/EFF
          RETURN
          END

C
C
C    SUBROUTINE STAT
C
          SUBROUTINE STAT(N,R,BX,DX,BOX,DOX)
          INTEGER J,R,N
          REAL BX(N+1),DX(N+1),BOX(R),DOX(R)
             BOX(2)= 0.0
             DOX(2)= 0.0
C
C    COMPUTE THE ESTIMATES OF THE SAMPLE MEAN AND
C   VARIANCE
C
             BOX(1) = BX(N+1)/N
             DOX(1) = DX(N+1)/N
             DO 260 I=1,N
                BOX(2)= BOX(2) + (BX(I)-BOX(1))**2
```

```
            DOX(2)= DOX(2) + (DX(I)-DOX(1))**2
260      CONTINUE
         BOX(2)= BOX(2)/(N*(N-1))
         DOX(2)= DOX(2)/(N*(N-1))
      RETURN
      END
```

APPENDIX C

FORTRAN PROGRAM LISTING OF THE BCD SIMULATION USING ANTITHETIC VARIATES

```
      DIMENSION BX(101),DX(101),SEED(2),BOX(2),DOX(2)
      INTEGER I,N,BB,DD,R
      REAL*4 X,Y,Z,BX,DX,TXT,BOX,DOX
      DOUBLE PRECISION SEED
      DATA SEED /1234.0D0,567890123.0D0/
      R= 2
      N=50
C
C  RECEIVE INPUT DATA FROM TERMINAL
C
      WRITE(*,3)
 3    FORMAT(1X,'ENTER THE NUMBER OF BOMBERS')
      READ '(I2)', BB
      WRITE(*,4)
 4    FORMAT(1X,'ENTER THE RATE WHICH A BOMBER SHOOTS'
     & 'DOWN A  DEFENDER')
      READ '(F5.3)', Y
      WRITE(*,5)
 5    FORMAT(1X,'ENTER THE NUMBER OF DEFENDERS')
      READ '(I2)', DD
      WRITE(*,6)
 6     FORMAT(1X,'ENTER THE RATE WHICH A DEFENDER
     & SHOOTS DOWN A BOMBER')
      READ '(F5.3)', X
      WRITE(*,7)'ENTER THE RATE WHICH  FREE DEFENDERS'
     & 'FIND FREE BOMBERS'
 7    FORMAT(1X,A)
      READ '(F5.3)', Z
      WRITE(*,8)'ENTER THE  TIME DURATION OF THE'
     & 'BATTLE'
 8    FORMAT(1X,A)
      READ '(F5.3)', TXT
C
C   RUN REPLICATIONS OF N BATTLES AND OBTAIN SUMMARY OF
C   N   BATTLES
C
      CALL BATTLE(N,R,SEED,X,Y,Z,BB,DD,TXT,BX,DX)
C
C  COMPUTE STATISTICAL OUTPUT ANALYSIS OBTAIN PARAMETER
C  ESTIMATES
C
      CALL STAT(N,R,BX,DX,BOX,DOX)
C
```

```
C  FORMAT AND PRINT OUTPUT OF PARAMETER ESTIMATES
C
       WRITE(3,279) N
 279   FORMAT(15X,'SAMPLE SIZE',I6,/15X,6('-'),
     & 1X,4('-'))
       WRITE(3,280) BB,DD
 280   FORMAT(//1X,I4,2X,'BOMBERS',6X,'VERSUS',
     & 6X,I4,2X,'DEFENDERS',/1X,13('-'),18X,14('-'))
 290   WRITE (3,300)TXT
 300   FORMAT(//18X,'TIME',F6.1,/,47('-'))
       WRITE(3,310)
 310   FORMAT(19X,'BOMBER',2X,'DEFENDER',
     & /18X,7('='),2X,8('='))
       WRITE(3,320) BOX(1),DOX(1)
 320   FORMAT(1X,'AVERAGE',7X,2F10.4)
       WRITE(3,330) BOX(2),DOX(2)
 330   FORMAT(1X,'VARIANCE',6X,2F10.4)
       STOP
       END
C
C
C   SUBROUTINE BATTLE
C
       SUBROUTINE
     & BATTLE(N,R,SEED,X,Y,Z,BB,DD,TXT,BX,DX)
       INTEGER BB,DD,H,I,J,W,K,N,R
       REAL GA(2,1000),SOJ(2,1000),BX(N+1),
     & DX(N+1),BAT(50,2),DAT(50,2),
     & X,Y,Z,T,TXT,TIME,INF,NC,BW,DW,B,C,D
       DOUBLE PRECISION SEED(R)
       K= 1000
C
C INITIALIZE STATISTICAL COUNTERS
C
       BX(N+1)= 0.0
       DX(N+1)= 0.0
C
C RUN N REPLICATIONS
C
       DO 200 I=1,N
          CALL SOJOUR(SEED(1),SOJ,R,K)
          CALL STATE(SEED(2),GA,R,K)
          DO 175 J=1,R
C
C   INITIALIZE START-TO-BATTLE VALUES
C
                 H=0
                 T=0.0
                 C=0.0
                 B=REAL(BB)
                 D=REAL(DD)
```

```
                  BW= 0.0
                  DW= 0.0
                  NC= 1.0
C
C  OBSERVE OCCURRENCE OF NEXT EVENT
C
 100              H=H+1
C
C DETERMINE NEXT INTERIM EVENT AND UPDATE THE STATE
C  (B,C,D)
C
                IF(GA(J,H) .LE. BW) THEN
                  B=B + 1.0
                  C=C - 1.0
                  D=D
                ELSE IF (GA(J,H) .LE. (BW+NC)) THEN
                  B=B - 1.0
                  C=C + 1.0
                  D=D - 1.0
                ELSE
                  B=B
                  C=C - 1.0
                  D=D + 1.0
                END IF
C
C  COMPUTE MEAN TIME IN STATE (B,C,D)
C
                IF ((B  .EQ. 0.0 .OR. D .EQ. 0.0) .AND. C
     &          .EQ. 0.0) THEN
                    INF= 1000000.0
                ELSE
                    INF= 1.0/(Z*D*B + (X + Y)*C)
                END IF
C
C  COMPUTE SOJOURN TIME IN STATE (B,C,D)
C
                  TIME= -INF * ALOG(SOJ(J,H))
C
C  ADVANCE THE SIMULATED TIME OF THE AIR BATTLE
C
                T= T + TIME
C
C  COMPUTE PROBABILITY OF NEXT INTERIM EVENTS OCCURRING

C
                  BW= Y*C*INF
                  DW= X*C*INF
                  NC= Z*B*D*INF
C
C  CHECK FOR OCCURRENCE OF TERMINATION EVENT
C
```

```
                IF (T .LT. TXT) GOTO 100
C
C   RECORD RESULTS OF BATTLE
C
                BAT(I,J)= B + C
                DAT(I,J)= D + C
 175        CONTINUE
C
C   ACCUMULATE SUMMARY OF N BATTLE OUTCOMES
C
            BX(I)=(BAT(I,1) + BAT(I,2))*.5
            DX(I)=(DAT(I,1) + DAT(I,2))*.5
            BX(N+1)= BX(N+1) + BX(I,J)
            DX(N+1)= DX(N+1) + DX(I,J)
 200     CONTINUE
         RETURN
         END
C
C
C   SUBROUTINE SOJOUR
C
         SUBROUTINE SOJOUR(SEED2,A2,W,K)
         INTEGER I,W,K
         REAL A2(W,K)
         DOUBLE PRECISION EFF,SEED2
         EFF= 2147483647.0D0
         DO 10 I=1,K
            SEED2=DMOD(16807.0D0 * SEED2,EFF)
            A2(1,I)= SEED2/EFF
            A2(2,I)= 1.0 - A2(1,I)
 10      CONTINUE
         RETURN
         END
C
C
C   SUBROUTINE STATE
C
         SUBROUTINE STATE(SEED1,A1,W,K)
         INTEGER I,K,W
         REAL A1(W,K)
         DOUBLE PRECISION EFF,SEED1
         EFF= 2147483647.0D0
         DO 10 I=1,K
            SEED1=DMOD(16807.0D0 * SEED1,EFF)
            A1(1,I)= SEED1/EFF
            A1(2,I)= 1.0 - A1(1,I)
 10      CONTINUE
         RETURN
         END
C
C
```

```
C   SUBROUTINE STAT
C
      SUBROUTINE STAT(N,R,BX,DX,BOX,DOX)
      INTEGER J,R,N
      REAL BX(N+1),DX(N+1),BOX(R),DOX(R)
         BOX(2)= 0.0
         DOX(2)= 0.0
C
C   COMPUTE THE ESTIMATES OF THE SAMPLE MEAN AND
C  VARIANCE
C
         BOX(1) = BX(N+1,J)/N
         DOX(1) = DX(N+1,J)/N
         DO 260 I=1,N
            BOX(2)= BOX(2) + (BX(I)-BOX(1))**2
            DOX(2)= DOX(2) + (DX(I)-DOX(1))**2
 260     CONTINUE
         BOX(2)= BOX(2)/(N*(N-1))
         DOX(2)= DOX(2)/(N*(N-1))
      RETURN
      END
```

APPENDIX D

FORTRAN PROGRAM LISTING OF THE BCD SIMULATION USING STRATIFIED SAMPLING

```
      DIMENSION BX(101),DX(101),SEED(2),BOX(2),DOX(2)
      INTEGER I,N,BB,DD,R
      REAL*4 X,Y,Z,BX,DX,TXT,BOX,DOX
      DOUBLE PRECISION SEED
      DATA SEED /1234.0D0,567890123.0D0/
      R= 2
      N=100
C
C  RECEIVE INPUT DATA FROM TERMINAL
C
      WRITE(*,3)
 3    FORMAT(1X,'ENTER THE NUMBER OF BOMBERS')
      READ '(I2)', BB
      WRITE(*,4)
 4    FORMAT(1X,'ENTER THE RATE WHICH A BOMBER SHOOTS'
     & 'DOWN A DEFENDER')
      READ '(F5.3)', Y
      WRITE(*,5)
 5    FORMAT(1X,'ENTER THE NUMBER OF DEFENDERS')
      READ '(I2)', DD
      WRITE(*,6)
 6     FORMAT(1X,'ENTER THE RATE WHICH A DEFENDER'
     & 'SHOOTS DOWN A BOMBER')
      READ '(F5.3)', X
      WRITE(*,7)'ENTER THE RATE WHICH  FREE DEFENDERS'
     & 'FIND FREE BOMBERS'
 7    FORMAT(1X,A)
      READ '(F5.3)', Z
      WRITE(*,8)'ENTER THE  TIME DURATION OF THE'
     & 'BATTLE'
 8    FORMAT(1X,A)
      READ '(F5.3)', TXT
C
C   RUN REPLICATIONS OF N BATTLES AND OBTAIN SUMMARY OF
C   N BATTLES
C
      CALL BATTLE(N,R,SEED,X,Y,Z,BB,DD,TXT,BX,DX)
C
C  COMPUTE STATISTICAL OUTPUT ANALYSIS OBTAIN PARAMETER
C  ESTIMATES
C
      CALL STAT(N,R,BX,DX,BOX,DOX)
```

```
C
C  FORMAT AND PRINT OUTPUT OF PARAMETER ESTIMATES
C
       WRITE(3,279) N
 279   FORMAT(15X,'SAMPLE SIZE',I6,/15X,6('-'),
     & 1X,4('-'))
       WRITE(3,280) BB,DD
 280        & FORMAT(//1X,I4,2X,'BOMBERS',6X,'VERSUS',
     & 6X,I4,2X,'DEFENDERS',
     & /1X,13('-'),18X,14('-'))
 290   WRITE (3,300)TXT
 300   FORMAT(//18X,'TIME',F6.1,/,47('-'))
       WRITE(3,310)
 310    FORMAT(19X,'BOMBER',2X,'DEFENDER',
     & /18X,7('='),2X,8('='))
       WRITE(3,320) BOX(1),DOX(1)
 320   FORMAT(1X,'AVERAGE',7X,2F10.4)
       WRITE(3,330) BOX(2),DOX(2)
 330   FORMAT(1X,'VARIANCE',6X,2F10.4)
       STOP
       END
C
C
C   SUBROUTINE BATTLE
C
       SUBROUTINE
     & BATTLE(N,R,SEED,X,Y,Z,BB,DD,TXT,BX,DX)
       INTEGER BB,DD,H,I,J,K,N,R,W
       REAL GA(3,1000),SOJ(3,1000),BM(34,3),
     & DF(34,3),BX(N+1),DX(N+1),X,Y,Z,T,TXT,TIME,
     & INF,NC,BW,DW,B,C,D,BM,DF
       DOUBLE PRECISION SEED(R)
       K= 1000
       W= 3
C
C INITIALIZE STATISTICAL COUNTERS
C
       BX(N+1)= 0.0
       DX(N+1)= 0.0
C
C RUN N REPLICATIONS
C
       DO 200 I=1,N
          CALL SOJOUR(SEED(1),SOJ,W,K)
          CALL STATE(SEED(2),GA,W,K)
          DO 175 J=1,W
C
C   INITIALIZE START-TO-BATTLE VALUES
C
                H=0
                T=0.0
```

```
                  C=0.0
                  B=REAL(BB)
                  D=REAL(DD)
                  NC= 1.0
                  BW= 0.0
                  DW= 0.0
C
C  OBSERVE AN OCCURRENCE OF AN EVENT
C
 100              H=H+1
C
C DETERMINE NEXT INTERIM EVENT AND UPDATE THE STATE
C     (B,C,D)
C
                IF(GA(J,H) .LE. BW) THEN
                  B=B + 1.0
                  C=C - 1.0
                  D=D
                ELSE IF (GA(J,H) .LE. (BW+NC)) THEN
                  B=B - 1.0
                  C=C + 1.0
                  D=D - 1.0
                ELSE
                  B=B
                  C=C - 1.0
                  D=D+ 1.0
                END IF
C
C  COMPUTE MEAN TIME IN STATE (B,C,D)
C
                IF((B .EQ.  0.0 .OR. D .EQ. 0.0).AND. C
     &           .EQ. 0.0) THEN
                   INF= 1000000.0
                ELSE
                   INF= 1.0/(Z*B*D + (X+Y)*C)
                END IF
C
C  COMPUTE SOJOURN TIME OF STATE (B,C,D)
C
                  TIME= -INF * ALOG(SOJ(J,H))
C
C  ADVANCE SIMULATED TIME OF THE AIR BATTLE
C
                T= T + TIME
C
C  COMPUTE PROBABILITY OF NEXT INTERIM EVENTS OCCURRING

C
                  NC= Z*B*D*INF
                  BW= Y*C*INF
                  DW= X*C*INF
```

```
C
C  CHECK FOR OCCURRENCE OF TERMINATION EVENT
C
              IF (T .LT. TXT(TI)) GOTO 100
C
C   RECORD RESULTS OF BATTLE
C
              BM(I,J)= B + C
              DF(I,J)= D + C
 175      CONTINUE
C
C   ACCUMULATE SUMMARY OF N BATTLE OUTCOMES
C
          BX(I)= (BM(I,1) + BM(I,2) + BM(I,3)/3.0
          DX(I)= (DF(I,1) + DF(I,2) + BM(I,3)/3.0
          BX(N+1)= BX(N+1) + BX(I)
          DX(N+1)= DX(N+1) + DX(I)
 200   CONTINUE
       RETURN
       END
C
C
C   SUBROUTINE SOJOUR
C
       SUBROUTINE SOJOUR(SEED2,B2,W,K)
       INTEGER I,W,K,J
       REAL B2(W,K),A2
       DOUBLE PRECISION EFF,SEED2
       EFF= 2147483647.0D0
       DO 10 I=1,K
          SEED2=DMOD(16807.0D0 * SEED2,EFF)
          A2= SEED2/EFF
          DO 5 J=1,W
             B2(J,I)= AMOD(A2 + ((J-1) * 1.0)/3.0,1.0)
 5        CONTINUE
 10    CONTINUE
       RETURN
       END
C
C
C   SUBROUTINE STATE
C
       SUBROUTINE STATE(SEED1,A1,W,K)
       INTEGER I,K,W,J
       REAL A1(W,K),A2
       DOUBLE PRECISION EFF,SEED1
       EFF= 2147483647.0D0
       DO 10 I=1,K
          SEED1=DMOD(16807.0D0 * SEED1,EFF)
          A2= SEED1/EFF
          DO 5 J=1,W
```

```
             A1(J,I)= AMOD(A2 + ((J-1) * 1.0)/3.0,1.0)
 5        CONTINUE
       RETURN
       END
C
C
C   SUBROUTINE STAT
C
       SUBROUTINE STAT(N,R,BX,DX,BOX,DOX)
       INTEGER R,N
       REAL BX(N+1),DX(N+1),BOX(R),DOX(R)
          BOX(2)= 0.0
          DOX(2)= 0.0
C
C   COMPUTE THE ESTIMATES OF THE SAMPLE MEAN AND
C  VARIANCE
C
          BOX(1) = BX(N+1)/N
          DOX(1) = DX(N+1)/N
          DO 260 I=1,N
             BOX(2)= BOX(2) + (BX(I)-BOX(1))**2
             DOX(2)= DOX(2) + (DX(I)-DOX(1))**2
 260      CONTINUE
          BOX(2)= BOX(2)/(N*(N-1))
          DOX(2)= DOX(2)/(N*(N-1))
       RETURN
       END
```

APPENDIX E

STATISTICAL OUTPUT DATA FROM THE BCD SIMULATIONS

TABLE E.1 OUTPUT PARAMETERS OF THE BCD MODEL ESTIMATED FROM CRUDE SIMULATION

	SCENARIO			DEFENDER		BOMBER	
SIMULATION	DEFENDER	BOMBER	TIME	MEAN	VAR	MEAN	VAR
1	10	10	25	7.6	.0191	7.5	.0250
2	10	10	75	4.4	.0387	4.5	.0358
3	10	10	125	2.8	.0375	3.4	.0417
4	10	30	25	5.3	.0328	25.1	.0541
5	10	30	75	1.3	.0157	21.3	.1402
6	10	30	125	.3	.0036	20.8	.1725
7	10	50	25	4.2	.0251	44.7	.0558
8	10	50	75	.6	.0064	40.8	.1654
9	10	50	125	.1	.0014	39.7	.1874
10	30	10	25	25.1	.0458	5.4	.0350
11	30	10	75	21.1	.1407	1.4	.0141
12	30	10	125	19.8	.2247	.4	.0045
13	30	30	25	18.7	.0933	18.5	.0842
14	30	30	75	8.2	.1618	8.4	.1895
15	30	30	125	5.7	.1984	4.7	.1341
16	30	50	25	14.4	.0628	34.5	.1104
17	30	50	75	3.4	.0362	22.9	.3479
18	30	50	125	.9	.0136	20.1	.4881
19	50	10	25	44.4	.0651	4.1	.0317
20	50	10	75	40.7	.1427	.7	.0063
21	50	10	125	41.1	.1666	.1	.0014
22	50	30	25	34.3	.1687	14.0	.0826
23	50	30	75	23.5	.3339	3.1	.0359
24	50	30	125	21.1	.5300	.8	.0107
25	50	50	25	27.9	.1590	26.4	.1501
26	50	50	75	9.5	.2717	11.5	.2865
27	50	50	125	5.2	.2126	7.9	.2986

TABLE E.2 OUTPUT PARAMETERS OF THE BCD MODEL ESTIMATED FROM THE SIMULATION USING ANTITHETIC VARIATES

	SCENARIO			DEFENDER		BOMBER	
SIMULATION	DEFENDER	BOMBER	TIME	MEAN	VAR	MEAN	VAR
1	10	10	25	7.6	.0067	7.5	.0105
2	10	10	75	4.5	.0048	4.4	.0060
3	10	10	125	3.2	.0079	3.3	.0057
4	10	30	25	5.1	.0129	25.1	.0100
5	10	30	75	1.2	.0071	21.3	.0329
6	10	30	125	.3	.0024	20.2	.0446
7	10	50	25	4.4	.0123	44.4	.0146
8	10	50	75	.6	.0060	40.9	.0395
9	10	50	125	.1	.0013	40.1	.0683
10	30	10	25	25.4	.0116	5.0	.0116
11	30	10	75	21.1	.0271	1.2	.0056
12	30	10	125	20.2	.0558	.3	.0030
13	30	30	25	18.5	.0164	18.4	.0210
14	30	30	75	8.5	.0251	8.4	.0270
15	30	30	125	5.1	.0307	5.1	.0352
16	30	50	25	14.6	.0223	34.6	.0308
17	30	50	75	3.0	.0138	23.6	.0542
18	30	50	125	1.0	.0075	20.6	.0446
19	50	10	25	44.0	.0169	4.3	.0104
20	50	10	75	40.2	.0410	.6	.0059
21	50	10	125	40.6	.0451	.1	.0015
22	50	30	25	34.3	.0279	14.4	.0368
23	50	30	75	23.1	.0375	3.1	.0194
24	50	30	125	20.5	.0739	.9	.0077
25	50	50	25	27.3	.0266	27.1	.0357
26	50	50	75	10.5	.0269	10.7	.0309
27	50	50	125	6.4	.0458	6.4	.0472

TABLE E.3 EFFICIENCY OF ANTITHETIC VARIATES IN THE BCD MODEL

	SCENARIO			DEFENDER		BOMBER	
SIMULATION	DEFENDER	BOMBER	TIME	RE	IP(%)	RE	IP(%)
1	10	10	25	2.9	65.0	2.4	58.0
2	10	10	75	8.1	87.6	6.0	83.2
3	10	10	125	4.7	78.9	7.3	86.3
4	10	30	25	2.5	60.7	5.4	81.5
5	10	30	75	2.2	54.8	4.3	76.5
6	10	30	125	1.5	33.3	3.9	74.1
7	10	50	25	2.0	51.0	3.8	73.8
8	10	50	75	1.1	6.3	4.2	76.1
9	10	50	125	1.1	7.1	2.7	63.6
10	30	10	25	3.9	74.7	3.0	66.9
11	30	10	75	5.2	80.7	2.5	60.3
12	30	10	125	4.0	75.2	1.5	33.3
13	30	30	25	5.7	82.4	4.0	75.1
14	30	30	75	6.4	84.5	7.0	85.8
15	30	30	125	6.5	84.5	3.8	73.8
16	30	50	25	2.8	64.5	3.6	72.1
17	30	50	75	2.6	61.9	6.4	84.4
18	30	50	125	1.8	44.9	10.9	90.9
19	50	10	25	3.9	74.0	3.0	67.2
20	50	10	75	3.5	71.2	1.1	6.3
21	50	10	125	3.7	72.9	.9	7.1
22	50	30	25	6.0	83.4	2.2	55.4
23	50	30	75	8.9	88.8	1.9	46.0
24	50	30	125	7.2	86.1	1.4	28.0
25	50	50	25	6.0	83.3	4.2	76.2
26	50	50	75	10.1	90.1	9.3	89.2
27	50	50	125	4.6	78.5	6.3	84.2

TABLE E.4 OUTPUT PARAMETERS OF THE BCD MODEL ESTIMATED FROM SIMULATION USING STRATIFIED SAMPLING

	SCENARIO			DEFENDER		BOMBER	
SIMULATION	DEFENDER	BOMBER	TIME	MEAN	VAR	MEAN	VAR
1	10	10	25	7.4	.0119	7.5	.0114
2	10	10	75	4.4	.0248	4.5	.0237
3	10	10	125	3.1	.0160	3.2	.0197
4	10	30	25	5.1	.0124	25.0	.0252
5	10	30	75	1.2	.0076	21.2	.0710
6	10	30	125	.4	.0038	20.5	.0801
7	10	50	25	4.4	.0126	44.0	.0375
8	10	50	75	.9	.0083	40.2	.1021
9	10	50	125	.1	.0006	40.0	.0930
10	30	10	25	25.1	.0233	5.3	.0124
11	30	10	75	21.1	.0726	1.3	.0118
12	30	10	125	20.3	.0877	.4	.0027
13	30	30	25	18.4	.0283	18.4	.0193
14	30	30	75	8.0	.0439	8.2	.0700
15	30	30	125	5.5	.0505	4.8	.0426
16	30	50	25	14.2	.0219	34.5	.0798
17	30	50	75	3.2	.0211	23.3	.1236
18	30	50	125	.8	.0081	20.1	.1880
19	50	10	25	44.6	.0339	4.3	.0163
20	50	10	75	41.0	.0872	.7	.0053
21	50	10	125	40.4	.1089	.1	.0012
22	50	30	25	34.6	.0571	14.3	.0439
23	50	30	75	22.6	.1187	3.3	.0197
24	50	30	125	20.9	.2402	1.0	.0086
25	50	50	25	27.7	.0630	27.0	.1084
26	50	50	75	10.5	.0898	10.7	.0881
27	50	50	125	6.5	.0911	6.4	.1084

TABLE E.5 EFFICIENCY OF STRATIFIED SAMPLING IN THE BCD MODEL

	SCENARIO			DEFENDER		BOMBER	
SIMULATION	DEFENDER	BOMBER	TIME	RE	IP(%)	RE	IP(%)
1	10	10	25	1.6	39.0	2.2	54.4
2	10	10	75	1.6	36.0	1.5	33.8
3	10	10	125	2.3	57.3	2.1	52.8
4	10	30	25	2.6	62.2	2.1	53.4
5	10	30	75	2.1	51.6	2.0	49.4
6	10	30	125	.9	-5.5	2.2	53.6
7	10	50	25	2.0	49.8	1.5	32.8
8	10	50	75	.8	-29.7	1.6	38.3
9	10	50	125	2.3	57.1	2.0	50.4
10	30	10	25	2.0	49.1	2.8	64.6
11	30	10	75	1.9	48.4	1.2	16.3
12	30	10	125	2.6	61.0	1.7	40.0
13	30	30	25	3.3	69.7	4.4	77.1
14	30	30	75	3.7	72.9	2.7	63.1
15	30	30	125	3.9	74.5	3.1	68.2
16	30	50	25	2.9	65.1	1.4	27.7
17	30	50	75	1.7	41.7	2.8	64.5
18	30	50	125	1.7	40.7	2.6	61.5
19	50	10	25	1.9	47.9	1.9	48.6
20	50	10	75	1.6	39.0	1.2	15.9
21	50	10	125	1.5	34.6	1.2	14.2
22	50	30	25	3.0	66.2	1.9	46.9
23	50	30	75	2.8	64.5	1.8	45.1
24	50	30	125	2.2	54.7	1.2	19.6
25	50	50	25	2.5	60.4	1.4	27.8
26	50	50	75	3.0	66.9	3.3	69.2
27	50	50	125	2.3	57.1	2.8	63.7

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